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Three-point functions in planar $\mathcal{N} = 4$ super Yang-Mills Theory for scalar operators up to length five at the one-loop order

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Abstract

We report on a systematic perturbative study of three-point functions in planar $SU(N)$ $\mathcal{N} = 4$ super Yang-Mills theory at the one-loop level involving scalar field operators up to length five. For this we have computed a sample of 40 structure constants involving primary operators of up to and including length five which are built entirely from scalar fields. A combinatorial dressing technique has been developed to promote tree-level correlators to one-loop level. In addition we have resolved the mixing up to the order g^2_{YM} level of the operators involved, which amounts to mixings with bi-fermions, with bi-derivative insertions as well as self-mixing contributions in the scalar sector. This work supersedes a preprint by two of the authors from 2010 which had neglected the mixing contributions.

Contents

1 Introduction and Conclusions

Following the discovery of integrable structures [1–3] in the AdS/CFT correspondence [4] our understanding of $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory [5] and the dual $AdS_5 \times S^5$ superstring theory has greatly advanced. To a large extent this progress occurred in the problem of finding the exact all-loop form of the anomalous scaling dimensions of local gauge invariant

operators of the gauge theory alias the spectrum of string excitations in the string model. The key was a mapping of the problem to an integrable spin chain which emerged from a one-loop perturbative study of the diagramatics involved by Minahan and Zarembo [1]. Moving on to higher loops the spectral problem was mapped to the diagonalization of a long-range spin chain model, whose precise microscopic form remains unknown [3, 6]. Nevertheless, assuming integrability the spin-chain S-matrix could be algebraically constructed and the spectral problem was rephrased for asymptotically long operators to the solution of a set of nested Bethe equation [7] (for reviews see [8–10]). The central remaining problem is now the understanding of wrapping interactions, which affect short operators at lower loop orders [11], [12]. From the algebraic viewpoint important progress was made by thermodynamic Bethe ansatz techniques [13] which also lead to a conjecture for the exact numerical scaling dimensions of the Konishi operator, the shortest unprotected operator in the theory [14].

Next to the scaling dimensions there also exist remarkable all-order results in planar $\mathcal{N} = 4$ SYM for supersymmetric Wilson-loops of special geometries [15] as well as for scattering amplitudes of four and five external particles [16], being closely related to light-like Wilson lines [17], see [18] for reviews.

Given these advances in finding exact results it is natural to ask if one can make similar statements for three-point functions of local gauge invariant operators. Due to conformal symmetry the new data appearing are the structure constants which have a nontrivial coupling constant $\lambda = g^2 N$ dependence and also appear in the associated operator product expansion. In detail we have for renormalized operators

$$\left\langle \widetilde{\mathcal{O}}_\alpha(x_1)\, \widetilde{\mathcal{O}}_\beta(x_2)\, \widetilde{\mathcal{O}}_\gamma(x_3) \right\rangle = \frac{C_{\alpha\beta\gamma}}{|x_{12}|^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma}|x_{23}|^{\Delta_\beta+\Delta_\gamma-\Delta_\alpha}|x_{13}|^{\Delta_\alpha+\Delta_\gamma-\Delta_\beta}|\mu|^{\gamma_\alpha+\gamma_\beta+\gamma_\gamma}}\,, \tag{1}$$

where $\Delta_\alpha = \Delta_\alpha^{(0)} + \lambda\,\gamma_\alpha$ denotes the scaling dimensions of the operators involved with $\Delta^{(0)}$ the engineering and γ the anomalous scaling dimensions, μ the renormalization scale and

$$C_{\alpha\beta\gamma} = C^{(0)}_{\alpha\beta\gamma} + \lambda\, C^{(1)}_{\alpha\beta\gamma} + O(\lambda^2) \tag{2}$$

is the scheme independent structure constant representing the new observable arising in three-point functions one would like to find. Similar to the case of two-point functions there are non-renormalization theorems for three-point correlation functions of chiral primary (or 1/2 BPS) operators, whose structure constants do not receive radiative corrections [19].

The study of three-point functions involving non-protected operators allowing for a non-trivial coupling constant dependence of the structure constants is still largely in its infancy. Direct computations of three-point functions are [20–26] while [27] analyzed the problem indirectly through an OPE decompostition of four-point functions of chiral primaries. The works [21, 26, 28] focused on non-extremal correlators involving scalar two-impurity operators which are particularly relevant in the BMN limit. The mixing problem of these operators with fermion and derivative impurities was analyzed in [29]. [22] considered extremal correlators of a very special class of operators allowing an interesting map to spin-chain correlation functions, while [30] addresses similar questions from the perspective of the non-planar contribution of the dilatation operator.

In the past year important advances in our understanding of the strong coupling behaviour of three-point functions were made based on the semi-classical analysis of the dual string theory. This approach to the calculation of n-point correlators involving non-BPS states was initiated in [31–34]. More recently, the authors of [32] argued that it should be possible to obtain the correlation functions of local operators corresponding to classical spinning string states, at strong coupling, by evaluating the string action on a classical solution with appropriate boundary conditions after convoluting with the classical states wavefunctions. In [33, 35], 2-point and 3-point correlators of vertex operators representing classical string states with large spins were calculated. Moreover, in a series of papers [34, 36] the 3-point function coefficients of correlators involving a massive string state, its conjugate and a third "light" state state were computed for a variety of massive string states. The result takes the form of a fattened Witten diagram with the vertex operator of the light state being integrated over the world-sheet of the classical solution describing the 2-point correlator of the heavy operator.

Furthermore, three-point functions of single trace operators were studied in [37–39] from the perspective of integrability. In particular, the protected $SU(2)$ scalar subsector of the $\mathcal{N}=4$ theory involving two holomorphic scalar fields was studied and upon exploting the underlying integrable spin chain structure analytic expressions for the tree-level piece $C^{(0)}_{\alpha\beta\gamma}$ could be established [37]. Recently these results were extended to the one-loop closed $SU(3)$ sector involving three holomorphic scalar fields [38]. In both cases limits of one short and two long operators led to serious simplifications. The one loop structure, however, has only been started to be explored in the $SU(2)$ subsector (see appendix E of [37]). Also an intriguing weak/strong coupling match of correlators involving operators in the $SU(3)$ sector was observed [38] [1]. This match was found to hold for correlators of two non-protected operators in the Frolov-Tseytlin limit and one short BPS operator. Subsequently, this weak/strong coupling match was extended for correlation functions involving operators in the $SL(2,R)$ closed subsector of $N=4$ SYM theory [41]. Finally, by performing Pohlmeyer reduction for classical solutions living in AdS_2 but with a prescribed nonzero energy-momentum tensor the authors of [42] calculated the AdS contribution to the three-point coefficient of three heavy states rotating purely in S^5 [2]. In the same spirit, part of the three-point fusion coefficient of three GKP strings [44] was calculated in [45][3] .

The two works [23, 24] considered the general problem of finding the structure constants of scalar field primary operators discussing important aspects of scheme independence for the determination of $C^{(1)}_{\alpha\beta\gamma}$.

In this paper we shall continue this work and report on a systematic one-loop study of short single trace conformal primary operators built from the six real scalar fields of the theory in the planar limit at the leading order. For this we developed a combinatorial dressing technique to promote tree-level non-extremal three-point correlation functions to the one-loop level which

[1] The authors of [40] have calculated the one-loop correction to the structure constants of operators in the $SU(2)$ sub-sector to find that this agreement is spoiled. However, there is the subtlety of the two "heavy" operators being *roughly* the conjugate of each other.

[2] The authors of [43] questioned this result arguing that string solutions with no AdS_5 charges should be point-like in the AdS_5 space.

[3] The contribution coming from the exact form of the vertex operators is still to be found.

is similar to the results reported in [24]. This is then used to compute a total of 40 structure constants at the one-loop level involving 11 different scalar field conformal primary operators up to length five. The restriction to this particular set of operators arose from the necessity to lift the operator degeneracy in the pure scalar $SO(6)$ sector by resolving the operator mixing problem arising from two-point correlators up to the two-loop. Indeed a large portion of our work is devoted to resolving the mixing problem of the scalar operators considered at the *two-loop* level. For this the results of [29, 26] for the mixing with bi-fermion and bi-derivative mixings have been extended to the two-loop self mixing sector as well as to two singlet operators of length four. In chapter 5 we spell out the form of the operators including all mixing corrections up to and including order $\mathcal{O}(\lambda^2)$. The main results of our work are collected in the tables 2, 3, 4 and 5 of section 6.

The main motivation for this spectroscopic study is to provide data to test and develop future conjectures on the form of the three-point structure constants most likely making use of integrability. It is important to stress that both results only apply for non-extremal correlation functions. Extremal correlation functions are such that $\Delta_\gamma^{(0)} = \Delta_\alpha^{(0)} + \Delta_\beta^{(0)}$ i.e. the length of the longest operator is equal to the sum of the two shorter ones. Here there also exists a proposed one-loop formula due to Okuyama and Tseng [23] see equation (27).

It would be very interesting to see whether these simple structures are stable at higher loop-order and also for non-purely scalar field primary operators such as the twist J operators for example.

2 General structure and scheme dependence of two and three-point functions

We want to compute planar two- and three-point functions of local scalar operators at the one-loop order. For this it is important to identify the regularization scheme independent information.

To begin with a scalar two-point function of bare local operators $\mathcal{O}^B_\alpha(x)$ in a random basis can be brought into diagonal form under a suitable linear transformation $\mathcal{O}_\alpha = M_{\alpha\beta}\mathcal{O}^B_\beta$ with a coupling constant $\lambda = g^2\,N$ independent mixing matrix $M_{\alpha\beta}$ as we are working at the one-loop level[4]

$$\langle \mathcal{O}_\alpha(x_1)\,\mathcal{O}_\beta(x_2)\rangle = \frac{\delta_{\alpha\beta}}{x_{12}^{2\Delta_\alpha^{(0)}}}\left(1+\lambda\,g_\alpha-\lambda\,\gamma_\alpha\,\ln|x_{12}\epsilon^{-1}|^2\right)\,, \qquad x_{12}^2 := (x_1-x_2)^2\,, \tag{3}$$

where ϵ represents a space-time UV-cutoff and $\Delta_\alpha^{(0)}$ the engineering scaling dimension of $\mathcal{O}_\alpha$. Clearly the finite contribution to the one-loop normalization g_α is scheme dependent [23, 24] as a shift in the cutoff parameter $\epsilon \to e^c\,\epsilon$ changes

$$g_\alpha \to g_\alpha + 2\,c\,\gamma_\alpha\,. \tag{4}$$

[4] Note that the two-loop diagonalization will involve a mixing matrix proportional to λ.

One may now define the renormalized operators via

$$\widetilde{\mathcal{O}}_\alpha = \mathcal{O}_\alpha \left(1 - \frac{\lambda}{2} g_\alpha - \lambda\gamma_\alpha \ln|\mu\,\epsilon| + O(\lambda^2)\right) \tag{5}$$

with a renormalization momentum scale μ to obtain finite canonical two-point correlation functions

$$\left\langle \widetilde{\mathcal{O}}_\alpha(x_1)\,\widetilde{\mathcal{O}}_\beta(x_2)\right\rangle = \frac{\delta_{\alpha\beta}}{|x_{12}|^{2\Delta_\alpha^{(0)}}}\left(1 - \lambda\gamma_\alpha \ln|x_{12}\mu|^2 + O(\lambda^2)\right) = \frac{\delta_{\alpha\beta}}{|x_{12}|^{2\Delta_\alpha^{(0)}}\,|x_{12}\mu|^{2\lambda\gamma_\alpha}}\,, \tag{6}$$

allowing one to extract the scheme independent one-loop scaling dimensions $\Delta_\alpha = \Delta_\alpha^{(0)} + \lambda\gamma_\alpha$.

Moving on to three-point functions of the un-renormalized diagonal operators $\mathcal{O}_\alpha$ one obtains to the one-loop order in λ

$$\begin{aligned}\langle\, \mathcal{O}_\alpha(x_1)\,\mathcal{O}_\beta(x_2)\,\mathcal{O}_\gamma(x_3)\,\rangle = {}& \frac{1}{|x_{12}|^{\Delta_\alpha^{(0)}+\Delta_\beta^{(0)}-\Delta_\gamma^{(0)}}\;|x_{23}|^{\Delta_\beta^{(0)}+\Delta_\gamma^{(0)}-\Delta_\alpha^{(0)}}\;|x_{31}|^{\Delta_\gamma^{(0)}+\Delta_\alpha^{(0)}-\Delta_\beta^{(0)}}} \\ & \times\left[C^{(0)}_{\alpha\beta\gamma}\left(1 + \frac{1}{2}\,\lambda\,\left\{\gamma_\alpha\,\ln\frac{\epsilon^2\,x_{23}^2}{x_{12}^2\,x_{31}^2} + \gamma_\beta\,\ln\frac{\epsilon^2\,x_{31}^2}{x_{12}^2\,x_{23}^2} + \gamma_\gamma\,\ln\frac{\epsilon^2\,x_{12}^2}{x_{23}^2\,x_{31}^2}\right\}\right) + \lambda\,\tilde{C}^{(1)}_{\alpha\beta\gamma}\right]\end{aligned} \tag{7}$$

Now again the finite one-loop contribution to the structure constant $\tilde{C}^{(1)}_{\alpha\beta\gamma}$ is scheme dependent [23, 24] as it changes under $\epsilon \to \epsilon\, e^c$ as

$$\tilde{C}^{(1)}_{\alpha\beta\gamma} \to \tilde{C}^{(1)}_{\alpha\beta\gamma} + c\,(\gamma_\alpha + \gamma_\beta + \gamma_\gamma)\,C^{(0)}_{\alpha\beta\gamma}\,, \qquad \text{(no sums on the indices)}\,. \tag{8}$$

However, the following combination of the unrenormalized three-point function structure constant and the normalization is scheme independent

$$C^{(1)}_{\alpha\beta\gamma} := \tilde{C}^{(1)}_{\alpha\beta\gamma} - \frac{1}{2}\,(g_\alpha\,C^{(0)}_{\alpha\beta\gamma} + g_\beta\,C^{(0)}_{\alpha\beta\gamma} + g_\gamma\,C^{(0)}_{\alpha\beta\gamma}\,)\,. \tag{9}$$

This is the only datum to be extracted from three-point functions. It also directly arises as the structure constant in the three-point function of the renormalized operators $\widetilde{\mathcal{O}}_\alpha$

$$\left\langle \widetilde{\mathcal{O}}_\alpha(x_1)\,\widetilde{\mathcal{O}}_\beta(x_2)\,\widetilde{\mathcal{O}}_\gamma(x_3)\right\rangle = \frac{C_{\alpha\beta\gamma}}{|x_{12}|^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma}|x_{23}|^{\Delta_\beta+\Delta_\gamma-\Delta_\alpha}|x_{13}|^{\Delta_\alpha+\Delta_\gamma-\Delta_\beta}|\mu|^{\lambda(\gamma_\alpha+\gamma_\beta+\gamma_\gamma)}}\,, \tag{10}$$

where $C_{\alpha\beta\gamma} = C^{(0)}_{\alpha\beta\gamma} + \lambda\,C^{(1)}_{\alpha\beta\gamma} + O(\lambda^2)$ is the scheme independent structure constant of (9).

An important further point is the following. If one wishes to compute the one-loop piece $C^{(1)}_{\alpha\beta\gamma}$ starting from a generic basis of operators one has to resolve the mixing problem at the two-loop order. This is so as the above discussed mixing matrix $M_{\alpha\beta}$ will receive $O(\lambda)$ terms once one computes the two-point function out to the order $O(\lambda^2)$. These mixing terms will contribute to the final $C^{(1)}_{\alpha\beta\gamma}$ through tree-level contractions. In this work we shall be interested in three-point

functions of single-trace operators which are given by purely scalar operators at leading order $\mathcal{O}_\alpha$. The mixing effects then induce a correction pattern of the schematic form

$$\hat{\mathcal{O}}_\alpha(g_{\rm YM},N) \sim {\rm Tr}(\phi^L)+g_{\rm YM}\,N\ {\rm Tr}(\psi\psi\phi^{L-3})+g_{\rm YM}^2\,N^2\ {\rm Tr}(D_\mu D_\mu\phi^{L-2})+g_{\rm YM}^2\,N\ {\rm Tr}(\phi^L)+\ldots\,, \quad (11)$$

where L is the length or engineering scaling dimension of $\mathcal{O}_\alpha$. Note that in the above schematic formula each trace-operator stands as a representative of a particular weighted combination of permutations of the same field content under the trace. Of course all operators transform in the same representation of the R-symmetry group $su(4)$ and are space-time scalars. We refer to the three-contributions as the fermionic $\mathcal{O}_{\psi\psi}$, the derivative $\mathcal{O}_{DD}$ and the self-mixing $\mathcal{O}_{\rm self}$ contributions. Clearly the tree-level insertions of these mixings, such as $\langle\mathcal{O}_{\alpha,0}\,\mathcal{O}_{\beta,\psi\psi}\,\mathcal{O}_{\beta,\psi\psi}\rangle$ or $\langle\mathcal{O}_{\alpha,0}\,\mathcal{O}_{\beta,0}\,\mathcal{O}_{\beta,DD}\rangle$ or $\langle\mathcal{O}_{\alpha,0}\,\mathcal{O}_{\beta,0}\,\mathcal{O}_{\beta,\rm self}\rangle$ contribute to the one-loop structure constants $C^{(1)}_{\alpha\beta\gamma}$ next to the radiative corrections discussed above. In addition the correlator $\langle\mathcal{O}_{\alpha,0}\,\mathcal{O}_{\beta,0}\,\mathcal{O}_{\beta,\psi\psi}\rangle$ with a Yukawa-vertex insertion will also contrbute potentially.

In our work we evaluate both these contributions – the radiative and mixing ones – and state the final result for the scheme independent structure constant $C^{(1)}_{\alpha\beta\gamma}$ for a large number of three-point functions. For this the mixing of scalar operators up to and including engineering length 5 has been determined.

3 The one-loop planar dressing formulae

3.1 Derivation

In this section we derive an efficient set of combinatorial dressing formulae to dress up tree-level graphs to one-loop. Similar formulae appeared in [46].

Following [21] we introduce the 4d propagator and the relevant one-loop integrals in configuration space

$$\begin{aligned}
I_{12} &= \frac{1}{(2\pi)^2 x_{12}^2}\,, \\
Y_{123} &= \int {\rm d}^4w\; I_{1w} I_{2w} I_{3w}\,, \\
X_{1234} &= \int {\rm d}^4w\; I_{1w} I_{2w} I_{3w} I_{4w}\,, \\
H_{12,34} &= \int {\rm d}^4v\,{\rm d}^4w\; I_{1v} I_{2v} I_{vw} I_{3w} I_{4w}\,, \\
F_{12,34} &= \frac{(\partial_1-\partial_2)\cdot(\partial_3-\partial_4) H_{12,34}}{I_{12} I_{34}}\,.
\end{aligned} \quad (12)$$

We have put the space-time points as indices to the function to make the expressions more compact. These functions are all finite except in certain limits. For example Y_{123} , X_{1234} and $H_{12,34}$ diverge logarithmically when $x_1 \to x_2$. In point splitting regularization one has the limiting

formulae ($\lim_{i\to j} x_{ij}^2 = \epsilon^2$)

$$X_{1123} = -\frac{1}{16\pi^2}\, I_{12} I_{13} \left(\ln \frac{x_{23}^2 \varepsilon^2}{x_{12}^2 x_{13}^2} - 2 \right), \tag{13}$$

$$Y_{112} = -\frac{1}{16\pi^2}\, I_{12} \left(\ln \frac{\varepsilon^2}{x_{12}^2} - 2 \right) = Y_{122}, \tag{14}$$

$$F_{12,13} = -\frac{1}{16\pi^2}\, \left(\ln \frac{\varepsilon^2}{x_{23}^2} - 2 \right) + Y_{123} \left(\frac{1}{I_{12}} + \frac{1}{I_{13}} - \frac{2}{I_{23}} \right), \tag{15}$$

$$X_{1122} = -\frac{1}{8\pi^2} I_{12}^2 \left(\ln \frac{\varepsilon^2}{x_{12}^2} - 1 \right), \tag{16}$$

$$F_{12,12} = -\frac{1}{8\pi^2}\, \left(\ln \frac{\varepsilon^2}{x_{12}^2} - 3 \right). \tag{17}$$

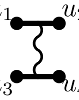
We introduce a graphical symbol for the scalar propagators and work in a normalization where

$$\left\langle \phi^I(x_1) \phi^J(x_2) \right\rangle_{\text{tree}} u_1^I u_2^J = \quad u_1 \quad u_2 \quad = (u_1 \cdot u_2)\, I_{12}\,, \tag{18}$$

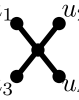
here the $SO(6)$-indices of the scalar fields are contracted with dummy six-vectors u_1^I and u_2^J for bookmarking purposes.

The one-loop corrections are then built of the following three components

$$u_1 \quad u_2 = -\lambda (u_1 \cdot u_2)\, I_{12}\, \frac{Y_{112} + Y_{122}}{I_{12}} \qquad \text{(self-energy)}, \tag{19}$$



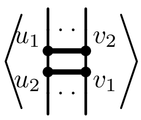
$$\begin{matrix} u_1 & u_2 \\ u_3 & u_4 \end{matrix} = \frac{\lambda}{2} (u_1 \cdot u_2)(u_3 \cdot u_4)\, I_{12}\, I_{34}\, F_{12,34} \qquad \text{(gluon)}, \tag{20}$$



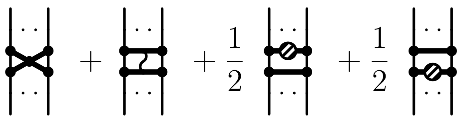
$$\begin{aligned} \begin{matrix} u_1 & u_2 \\ u_3 & u_4 \end{matrix} &= \frac{\lambda}{2} \big[2 (u_2 \cdot u_3)(u_1 \cdot u_4) - (u_2 \cdot u_4)(u_1 \cdot u_3) \\ &\qquad - (u_1 \cdot u_2)(u_3 \cdot u_4) \big]\, X_{1234} \qquad \text{(vertex)}. \end{aligned} \tag{21}$$

With these basic interactions we can now diagrammatically dress up the tree-level two- and three-point correlation functions to the one-loop level. To do so we note that a generic planar three-point function will be made of two-gon and three-gon sub-graphs which need to be dressed, see figure 1.

For the two-gon dressing one finds the basic dressing formula



$$= \quad + \quad + \frac{1}{2} \quad + \frac{1}{2}$$



$$= I_{12}^2\, \frac{\lambda}{8\pi^2} \left(\ln \frac{\epsilon^2}{x_{12}^2} - 1 \right) \left(u_1 \cdot v_2\, v_1 \cdot u_2 - u_1 \cdot u_2\, v_1 \cdot v_2 - \frac{1}{2}\, u_1 \cdot v_1\, u_2 \cdot v_2 \right)$$

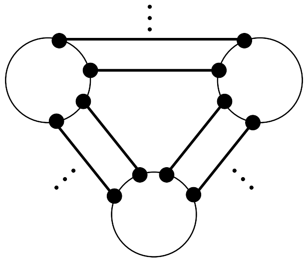
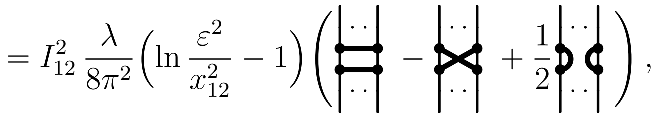


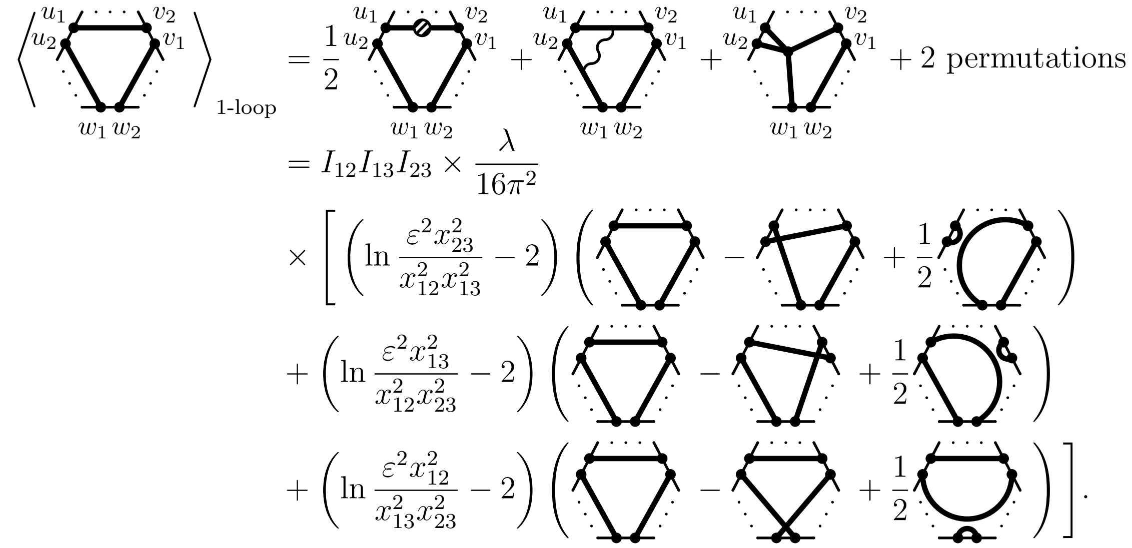
Figure 1: The generic tree-level three-point function.



(22)

where the diagrams in the last line only stand for the index contractions not for propagators. This contraction structure is of course that of an integrable nearest neighbor $SO(6)$ vector spin-chain Hamiltonian as was first noted in [1].

Analogously, for the three-gon we find



(23)

Again the graphs in the last three lines only represent the index contractions. Interestingly a similar structure to the integrable spin-chain Hamiltonian of (22) emerges also for the one-loop three-gon interactions.

3.2 Gauge invariance and Wilson line contributions

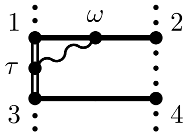
There is one important point we have not addressed so far. The point splitting regularization method that we employed violates gauge invariance as the space-time locations of the two neighboring operators in the trace are no longer coincident. The natural way to recover gauge

invariance is to connect the two split points through a straight Wilson line. This, however, gives rise to new diagrams not yet accounted for in which a gluon is radiated off the Wilson line. Luckily we are able to show that this contribution vanishes entirely at the one-loop level for $|\epsilon| \to 0$.

Setting $\epsilon^\mu = x_{13}^\mu$ the Wilson line is parametrized by

$$x^\mu(\tau) = x_3^\mu + \epsilon^\mu \tau\,, \quad \tau \in [0,1]\,. \tag{24}$$

We then have the contribution



$$= \lambda(u_1 \cdot u_2)(u_3 \cdot u_4) \int_0^1 d\tau\, \epsilon \cdot (\partial_1 - \partial_2)\, Y_{12\tau}$$

$$= -\frac{2\lambda(u_1 \cdot u_2)(u_3 \cdot u_4)}{(2\pi)^6} \int_0^1 d\tau \int d^4\omega\, \frac{\epsilon \cdot x_{1\omega}}{(x_{1\omega}^2)^2\, x_{2\omega}^2\, x_{\tau\omega}^2}\,. \tag{25}$$

This five dimensional integral is by power-counting logarithmically divergent for coincident points $x_3, x(\tau) \to x_1$ i.e. $|\epsilon| \to 0$ and one has

$$\lim_{|\epsilon|\to 0} \int_0^1 d\tau \int d^4\omega\, \frac{\epsilon \cdot x_{1\omega}}{(x_{1\omega}^2)^2\, x_{2\omega}^2\, x_{\tau\omega}^2} \sim \lim_{|\epsilon|\to 0} \epsilon \cdot x_{12} \left(\ln \frac{\epsilon^2}{x_{12}^2} + \text{finite} + O(\epsilon) \right) \to 0\,. \tag{26}$$

There is also a novel ladder-diagram in which a gluon is exchanged between two Wilson lines extending from x_1 to x_3 and from x_2 to x_4. This ladder-graph is manifestly finite and vanishes as ϵ^2. Therefore all the Wilson line contributions to the point splitting regularization vanish at this order of perturbation theory.

3.3 Extremal three-point functions

Three-point functions of operators with lengths $\varDelta_\alpha^{(0)}$, $\varDelta_\beta^{(0)}$ and $\varDelta_\gamma^{(0)}$ where $\varDelta_\alpha^{(0)} + \varDelta_\beta^{(0)} = \varDelta_\gamma^{(0)}$ are called extremal. For these extremal functions the dressing formulae above do not hold any longer for two reasons: First, there appear additional diagrams with a gluon exchange or a vertex between non-neighboring propagators as the one in figure 2. These non-nearest neighbor interactions lead to additional terms in the dressing formulae. Second, unlike non-extremal ones extremal three-point functions with double-trace operators contain the same factor of N as those with single-trace operators. This results in an operator mixing of single-trace with double-trace operators already at tree-level. This is described in detail in [47, 23].

We will refrain from studying these extremal three-point correlators in the following. In any case the one-loop structure constants follow a simple pattern: They are a given by a linear function of the anomalous scaling dimensions of the operators involved [23]

$$C_{\alpha\beta\gamma,\,\text{extremal}}^{(1)}\Big|_{\text{loop}} = \frac{1}{2}\, C_{\alpha\beta\gamma,\,\text{extremal}}^{(0)} \left(\gamma_\alpha + \gamma_\beta - \gamma_\gamma\right)\,, \tag{27}$$

hence the three-point problem has been reduced to the two-point one. In particular structure constants of protected operators are free of radiative corrections. However, the crucial mixing contributions are not taken into account in this formula which will also involve mixings with multi-trace operators.

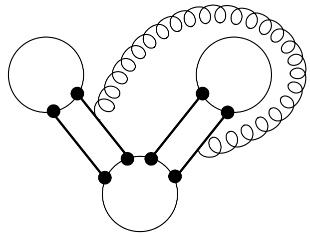


Figure 2: Additional Feynman-Graphs for extremal three-point functions.

3.4 Two convenient regularization schemes

We have seen in (9) how to extract the regularization scheme independent structure constant from a combination of the bare structure constant and the one-loop finite normalization shifts. As the latter arises from the finite contribution to the two-gon dressing (22) one may pick a regularization to simply cancel these contributions. I.e. making the transformation on the point-splitting parameter

$$\epsilon \to \sqrt{e}\,\epsilon \tag{28}$$

transforms

$$\ln\frac{\varepsilon^2}{x_{ij}^2} - 1 \to \ln\frac{\varepsilon^2}{x_{ij}^2}\,, \qquad \text{and} \qquad \ln\frac{\varepsilon^2 x_{ij}^2}{x_{ik}^2 x_{jk}^2} - 2 \to \ln\frac{\varepsilon^2 x_{ij}^2}{x_{ik}^2 x_{jk}^2} - 1\;. \tag{29}$$

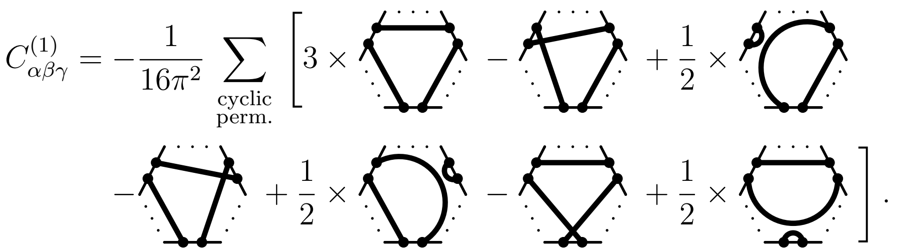
Hence in this scheme the finite part of the two-gon dressing vanishes resulting in a vanishing finite correction to the two-point functions

$$g_\alpha = 0\,, \tag{30}$$

which in turn implies that the bare and the renormalized structure functions coincide in this scheme

$$\widetilde{C}^{(1)}_{\alpha\beta\gamma} = C^{(1)}_{\alpha\beta\gamma}\,. \tag{31}$$

This implies that the structure function may be read off solely from the three-gon dressings of the non-extremal correlator, which may be graphically represented by



(32)

Alternatively one may apply the transformation

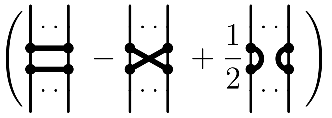
$$\epsilon \to e\,\epsilon \tag{33}$$

yielding

$$\ln\frac{\varepsilon^2}{x_{ij}^2} - 1 \to \ln\frac{\varepsilon^2}{x_{ij}^2} + 1\,, \qquad \text{and} \qquad \ln\frac{\varepsilon^2 x_{ij}^2}{x_{ik}^2 x_{jk}^2} - 2 \to \ln\frac{\varepsilon^2 x_{ij}^2}{x_{ik}^2 x_{jk}^2}\,. \tag{34}$$

Now the finite contributions to the three-gon dressings vanish and the bare structure constant may be computed from only dressing the two-gons in the tree-level correlator

$$\widetilde{C}^{(1)}_{\alpha\beta\gamma} = \frac{1}{8\pi^2}\sum_{\substack{\text{cyclic}\\ \text{perm.}}}\ \sum_{\substack{\text{all}\\ \text{2-gons}}} \tag{35}$$



The scheme independent structure constants can then be calculated using (9) with $g_\alpha = \gamma_\alpha$ by virtue of (34), i.e.

$$C^{(1)}_{\alpha\beta\gamma} = \widetilde{C}^{(1)}_{\alpha\beta\gamma} - \frac{1}{2}\, C^{(0)}_{\alpha\beta\gamma}\,\left(\gamma_\alpha + \gamma_\beta + \gamma_\gamma\right). \tag{36}$$

In our actual computations we have used both schemes depending on the problem at hand.

4 The considered operators and their mixing

As discussed above we also have to face the subtle problem of operator mixing. In this paper we aim at computing the structure constants of space-time scalar operators constructed exclusively from the $SO(6)$ scalar fields of $\mathcal{N} = 4$ super Yang-Mills

$$\mathcal{O}_\alpha = \sum c_{i_1\ldots i_L}\ \mathrm{Tr}(\phi_{i_1}\ldots\phi_{i_L})\,. \tag{37}$$

Due to operator mixing these will be corrected perturbatively by operators with bi-fermion and bi-derivative insertions, as well as 'self-mixings' in the purely scalar sector, i.e. we are facing the mixing structure

$$\begin{aligned}
\hat{\mathcal{O}}_\alpha &= \mathcal{O}_\alpha + g_{\mathrm{YM}}\,N\,\mathcal{O}_{\alpha,\psi\psi} + g_{\mathrm{YM}}^2\,N^2\,\mathcal{O}_{\alpha,DD} + g_{\mathrm{YM}}^2\,N\,\mathcal{O}_{\alpha,\mathrm{self}} + \mathcal{O}(g_{\mathrm{YM}}^3)\\
&\text{with}\\
\mathcal{O}_{\alpha,\psi\psi} &= \sum d_{i_1\ldots i_{L-3}}\ \mathrm{Tr}(\psi^\alpha\phi_{i_1}\ldots\phi_{i_*-1}\psi_\alpha\phi_{i_*}\ldots\phi_{i_{L-3}})\\
\mathcal{O}_{\alpha,DD} &= \sum e_{i_1\ldots i_{L-3};kl}\ \mathrm{Tr}(D^\mu\phi_{i_k}\phi_{i_1}\ldots\phi_{i_*-1}D_\mu\phi_{i_l}\phi_{i_*}\ldots\phi_{i_{L-4}})\\
\mathcal{O}_{\alpha,\mathrm{self}} &= \sum f_{i_1\ldots i_L}\ \mathrm{Tr}(\phi_{i_1}\ldots\phi_{i_L})\,.
\end{aligned} \tag{38}$$

These mixings will lead to contributions to the one-loop structure constant $C^{(1)}_{\alpha\beta\gamma}$ beyond the radiative ones discussed in the previous chapter through the tree-level correlators

$$\langle\mathcal{O}_{\alpha,\psi\psi}\,\mathcal{O}_{\beta,\psi\psi}\,\mathcal{O}_\gamma\rangle_0\,,\quad \langle\mathcal{O}_{\alpha,DD}\,\mathcal{O}_\beta\,\mathcal{O}_\gamma\rangle_0\,,\quad \langle\mathcal{O}_{\alpha,\mathrm{self}}\,\mathcal{O}_\beta\,\mathcal{O}_\gamma\rangle_0\,, \tag{39}$$

and their permutations, as well as the insertion of a Yukawa-vertex into the correlators

$$\langle\mathcal{O}_{\alpha,\psi\psi}\,\mathcal{O}_\beta\,\mathcal{O}_\gamma\rangle_{\mathrm{Yukawa}}\,, \tag{40}$$

which are all of order $\mathcal{O}(g^2_{\rm YM})$.

The operators we shall be considering in the non-protected sector are at leading order ($\varPhi_{AB} = \frac{1}{2}\,\epsilon_{ABCD}\,\varPhi^{CD}$ and $Z = \frac{1}{\sqrt{2}}\,(\phi_5 + i\phi_6)$ are the complexified scalars, see appendix A for conventions)

$$\mathcal{O}_{2A} = \mathcal{K} = \frac{8\pi^2}{\sqrt{3}N}\mathrm{Tr}\left(\varPhi_{AB}\varPhi^{AB}\right) \tag{41}$$

$$\mathcal{O}^J_n = \sqrt{\frac{(8\pi^2)^{J+2}}{N^{J+2}(J+3)}}\Big\{\sum_{p=0}^{J}\cos\frac{\pi n(2p+3)}{J+3}\mathrm{Tr}\left(\varPhi_{AB}Z^p\varPhi^{AB}Z^{J-p}\right)\Big\}$$
$$J = 1,2,3 \qquad n = 1,\ldots,[J/2] \tag{42}$$

$$\hat{\mathcal{O}}_{4A/E} = \left(\frac{8\pi^2}{N}\right)^2\frac{4}{\sqrt{738\mp 102\sqrt{41}}}\Big\{\mathrm{Tr}\left(\varPhi_{AB}\varPhi^{AB}\varPhi_{CD}\varPhi^{CD}\right) + \frac{5\mp\sqrt{41}}{4}\mathrm{Tr}\left(\varPhi_{AB}\varPhi_{CD}\varPhi^{AB}\varPhi^{CD}\right)\Big\} \tag{43}$$

From the mixing pattern in (38) we see that the length two Konishi operator $\mathcal{O}_{2A} = \mathcal{K}$ is protected from mixing. The operators $\mathcal{O}^J_n$ constitute a whole family of $SO(4)\times SO(4)$ single trace singlets with classical conformal dimension $\varDelta^{(0)} = J+2$, where J is the charge under a $U(1)\in SO(6)$ under which the Z fields carry charge 1. These states belong to the $[0,J,0]$ representation of $SU(4)_R$ and they are the highest weights of a long representation of the $PSU(2,2|4)$ superconformal algebra. For each value of J, there are $E\left[\frac{J+2}{2}\right]$ such eigenstates of the planar two-loop conformal dimension, labeled by an integer number $1\le n\le E\left[\frac{J+2}{2}\right]$. Indeed, the bi-fermion and bi-derivative mixings of these operators have been worked out in [29, 26]:

$$\begin{aligned}
\hat{\mathcal{O}}^J_n =\mathcal{N}\Bigg\{&\sum_{p=0}^{J}\cos\frac{\pi n(2p+3)}{J+3}\mathrm{Tr}\left(\varPhi_{AB}Z^p\varPhi^{AB}Z^{J-p}\right)\\
&g\frac{N}{8\sqrt{2}\pi^2}\sin\frac{\pi n}{J+3}\sum_{p=0}^{J-1}\sin\frac{\pi n(2p+4)}{J+3}\mathrm{Tr}\left(\psi^{1\alpha}Z^p\psi^2_\alpha Z^{J-p-1}\right)\\
&-g\frac{N}{8\sqrt{2}\pi^2}\sin\frac{\pi n}{J+3}\sum_{p=0}^{J-1}\sin\frac{\pi n(2p+4)}{J+3}\mathrm{Tr}\left(\bar\psi_{3\dot\alpha}Z^p\bar\psi^{\dot\alpha}_4 Z^{J-p-1}\right)\\
&+g^2\frac{N^2}{(8\sqrt{2}\pi^2)^2}\sin^2\frac{\pi n}{J+3}\sum_{p=0}^{J-2}\cos\frac{\pi n(2p+5)}{J+3}\mathrm{Tr}\left(D_\mu ZZ^pD^\mu ZZ^{J-p-2}\right)\\
&+g^2\sum_{\substack{m=1\\m\neq n}}^{E\left[\frac{J+2}{2}\right]}\mathcal{C}^{\rm self}_{n,m}\sum_{p=0}^{J}\cos\frac{\pi m(2p+3)}{J+3}\mathrm{Tr}\left(\varPhi_{AB}Z^p\varPhi^{AB}Z^{J-p}\right)\Bigg\},
\end{aligned} \tag{44}$$

where the self-mixing coefficient $\mathcal{C}^{\rm self}_{n,m}$ is still undetermined and the normalization up to one loop

is (see the computation in appendix B)

$$\mathcal{N} = \sqrt{\frac{N_0^{-J-2}}{J+3}} \Big[1 + g_{YM}^2 N \frac{\sin^2 \frac{\pi n}{J+3}}{2\pi^2(J+3)} \left(\frac{J-1}{2} + 2\cos^2 \frac{2\pi n}{J+3}\right) - \frac{g_{YM}^2 N}{2} g_{\mathcal{O}_n^J} \\ + g_{YM}^2 N \gamma_{\mathcal{O}_n^J} \ln \left|\frac{\Lambda}{\mu}\right| + \mathcal{O}(g_{YM}^4)\Big], \tag{45}$$

where $N_0 = \frac{N}{8\pi^2}$ and $g_{\mathcal{O}_n^J}$ is the scheme dependent finite one-loop contribution discussed in section 2. The mixing with the terms containing fermionic and derivative impurities in (44) has been computed in [29, 26] by requiring that the operator is annihilated by the superconformal charges up to one-loop. In the next section we shall compute the coefficient of the self-mixing in the last line of (44), by requiring that the full operator $\hat{\mathcal{O}}_n^J$ is an eigenstate of the dilatation operator at two loops constructed in [48]. As we shall be studying only correlators involving operators up to engineering lenth 5, we shall be studying the set of operators

$$\{\, \mathcal{O}_{4F} := \mathcal{O}_1^2,\ \mathcal{O}_{4B} := \mathcal{O}_2^2,\ \mathcal{O}_{5J} := \mathcal{O}_1^3,\ \mathcal{O}_{5E} := \mathcal{O}_2^3 \,\} \tag{46}$$

in this $\hat{\mathcal{O}}_n^J$ operator family. We also often use the alternative (historic) nomenclature pattern introduced in the preprint [49]. Finally we note that the one-loop scaling dimension of $\hat{\mathcal{O}}_n^J$ reads $\gamma_{Jn} = 4\lambda \sin^2 \frac{\pi n}{J+3}$.

4.1 Self-mixing contributions from H_4

With $a, b, c, d, e, f, i, j, k = 1, \ldots, 6$ denoting the $SO(6)$ indices of the scalar fields Φ_i one may write the pure $SO(6)$ piece of the one and two-loop hamiltonians H_2 and H_4 as follows [1, 6, 48]

$$H_2 = \frac{\lambda}{4}\left(2\begin{Bmatrix} ab \\ ab \end{Bmatrix} - 2\begin{Bmatrix} ba \\ ab \end{Bmatrix} + \begin{Bmatrix} bb \\ aa \end{Bmatrix}\right), \qquad \lambda := \frac{g_{\mathrm{YM}}^2 N}{4\pi^2}\,, \tag{47}$$

$$\begin{aligned} H_4 = \frac{\lambda^2}{4}\Big(&-2\begin{Bmatrix} abc \\ abc \end{Bmatrix} + \frac{3}{2}\Big[\begin{Bmatrix} bac \\ abc \end{Bmatrix} + \begin{Bmatrix} acb \\ abc \end{Bmatrix}\Big] - \frac{1}{2}\Big[\begin{Bmatrix} bca \\ abc \end{Bmatrix} + \begin{Bmatrix} cab \\ abc \end{Bmatrix}\Big] \\ &- \frac{11}{16}\Big[\begin{Bmatrix} bbc \\ aac \end{Bmatrix} + \begin{Bmatrix} cbb \\ caa \end{Bmatrix}\Big] + \frac{1}{4}\begin{Bmatrix} bcb \\ aca \end{Bmatrix} - \frac{1}{16}\Big[\begin{Bmatrix} bbc \\ caa \end{Bmatrix} + \begin{Bmatrix} cbb \\ aac \end{Bmatrix}\Big] \\ &+ \frac{1}{8}\Big[\begin{Bmatrix} bbc \\ aca \end{Bmatrix} + \begin{Bmatrix} cbb \\ aca \end{Bmatrix} + \begin{Bmatrix} bcb \\ aac \end{Bmatrix} + \begin{Bmatrix} bcb \\ caa \end{Bmatrix}\Big]. \end{aligned} \tag{48}$$

Where the action of theses operators is defined by

$$\begin{Bmatrix} abc \\ def \end{Bmatrix} |\Phi_i\, \Phi_j \Phi_k\rangle = \delta_{di}\, \delta_{ej}\, \delta_{fk}\, |\Phi_a\, \Phi_b \Phi_c\rangle\,, \tag{49}$$

and repeated indices are summed over. Note that we are now employing a spin-chain language representing a local gauge invariant operator by a state.

We now use this result in order to determine the scalar contributions to the self-mixing coefficients in (44). Upon acting on H_4 one finds for the operators of (42) and (43)

$$
\begin{aligned}
H_4 \circ \mathcal{O}_1^2 &= \frac{-19+7\sqrt{5}}{8}\lambda^2\,\mathcal{O}_1^2 + \frac{-2}{4}\lambda^2\,\mathcal{O}_2^2\,,\\
H_4 \circ \mathcal{O}_2^2 &= \frac{-2}{4}\lambda^2\,\mathcal{O}_1^2 + \frac{-19-7\sqrt{5}}{8}\lambda^2\,\mathcal{O}_2^2\,,\\
H_4 \circ \mathcal{O}_1^3 &= -\frac{1}{8}\lambda^2\,\mathcal{O}_1^3 - \frac{\sqrt{3}}{8}\lambda^2\,\mathcal{O}_2^3\,,\\
H_4 \circ \mathcal{O}_2^3 &= -\frac{\sqrt{3}}{8}\lambda^2\,\mathcal{O}_1^3 - \frac{27}{8}\lambda^2\,\mathcal{O}_2^3\,,\\
H_4 \circ \mathcal{O}_{4A} &= (-\frac{25}{8} - \frac{181}{8\sqrt{41}})\lambda^2\,\mathcal{O}_{4A} - \frac{\sqrt{5}}{2\sqrt{41}}\lambda^2\,\mathcal{O}_{4E}\,,\\
H_4 \circ \mathcal{O}_{4E} &= -\frac{\sqrt{5}}{2\sqrt{41}}\lambda^2\,\mathcal{O}_{4A} + (-\frac{25}{8} + \frac{181}{8\sqrt{41}})\lambda^2\,\mathcal{O}_{4E}\,.
\end{aligned}
\tag{50}
$$

From this one deduces the following corrected two-loop eigenstates using standard non-degenerate perturbation theory

$$\hat{\mathcal{O}}_1^2 = \mathcal{O}_1^2 + \lambda\,\frac{1}{2\sqrt{5}}\,\mathcal{O}_2^2 = \hat{\mathcal{O}}_{4F}\,, \tag{51}$$

$$\hat{\mathcal{O}}_2^2 = \mathcal{O}_2^2 - \lambda\,\frac{1}{2\sqrt{5}}\,\mathcal{O}_1^2 = \hat{\mathcal{O}}_{4B}\,, \tag{52}$$

$$\hat{\mathcal{O}}_1^3 = \mathcal{O}_1^3 + \lambda\,\frac{\sqrt{3}}{16}\,\mathcal{O}_2^3 = \hat{\mathcal{O}}_{5J}\,, \tag{53}$$

$$\hat{\mathcal{O}}_2^3 = \mathcal{O}_2^3 - \lambda\,\frac{\sqrt{3}}{16}\,\mathcal{O}_1^3 = \hat{\mathcal{O}}_{5E}\,, \tag{54}$$

$$\hat{\mathcal{O}}_{4A} = \mathcal{O}_{4A} - \lambda\,\frac{\sqrt{5}}{41}\,\mathcal{O}_{4E}\,, \tag{55}$$

$$\hat{\mathcal{O}}_{4E} = \mathcal{O}_{4E} + \lambda\,\frac{\sqrt{5}}{41}\,\mathcal{O}_{4A}\,. \tag{56}$$

Note, however, that the self-mixing coefficients appearing in (44) receive additional contributions from the interactions with fermions appearing in H_3.

4.2 Self mixing contributions originating from H_3

We now evaluate the additional contributions to the self-mixing coefficients appearing in (44) originating from the interactions with fermions.

As discussed in [48] the self mixing has two sources. One is the 2-loop purely scalar Hamiltonian H_4 and the other is H_3. The total contributrion to the self-mixing is given by

$$\tilde{\mathcal{O}}_\alpha = \mathcal{O}_\alpha + g_{\rm YM}\,\mathcal{O}_{\alpha,\psi\psi} + g_{\rm YM}^2\,\mathcal{O}_{\alpha,DD} + g_{\rm YM}^2\sum_{\beta\neq\alpha}\frac{\langle\mathcal{O}_\beta|H_3|\mathcal{O}_{\alpha,\psi\psi}\rangle + \langle\mathcal{O}_\beta|H_4|\mathcal{O}_\alpha\rangle}{E_{2.\alpha}-E_{2,\beta}}\,\mathcal{O}_\beta\,. \tag{57}$$

The 2-loop energy of the state (57) is

$$E_{4,\alpha} = \langle \mathcal{O}_\alpha | H_4 | \mathcal{O}_\alpha \rangle + \langle \mathcal{O}_\alpha | H_3 | \mathcal{O}_{\alpha,\psi\psi} \rangle \qquad \text{where} \quad E_\alpha = \lambda\, E_{2,\alpha} + \lambda^2\, E_{4,\alpha} + \dots . \tag{58}$$

Before resolving the mixing for the specific states we are interested in, let us calculate via (58) the 2-loop eigenvalue of the operator O_1^2. The form of this class of operators is given up to order $g_{\rm YM}$ follows from the first three lines of (44). The corresponding spin chain state is obtained by performing the following substitutions [50]

$$\left(\frac{8\pi^2}{N}\right)^{\frac{1}{2}} Z_{YM} \to Z_{sp}, \qquad \left(\frac{8\pi^2}{N}\right)^{\frac{1}{2}} \frac{1}{\sqrt{2}} \psi^A_{YM} \to \psi^A_{sp} \tag{59}$$

This correspondence is such that $\langle \bar{Z}_i | Z_i \rangle = 1$ and similarly for the fermions. To simplify notation we will drop the subscript 'sp'. Consequently, the first three lines of eqn. (44) can be written in the spin chain language as the state

$$\begin{aligned}
|\hat{\mathcal{O}}^J_n\rangle = |\mathcal{O}^J_n\rangle + g_{\rm YM}\, |\mathcal{O}^J_{n,\psi\psi}\rangle &= \frac{1}{\sqrt{J+3}} \sum_{p=0}^{J} \cos\frac{\pi n(2p+3)}{J+3} |Z_i Z^p \bar{Z}_i Z^{J-p}\rangle \\
+ g_{\rm YM} \frac{\sqrt{N}}{4\pi} \frac{1}{\sqrt{J+3}} \sin\frac{\pi n}{J+3} & \sum_{p=0}^{J-1} \sin\frac{\pi n(2p+4)}{J+3}\, 2\, |\psi^{1\alpha} Z^p \psi^2_\alpha Z^{J-p-1}\rangle \\
- g_{\rm YM} \frac{\sqrt{N}}{4\pi} \frac{1}{\sqrt{J+3}} \sin\frac{\pi n}{J+3} & \sum_{p=0}^{J-1} \sin\frac{\pi n(2p+4)}{J+3}\, 2\, |\bar{\psi}_{3\dot{\alpha}} Z^p \bar{\psi}^{\dot{\alpha}}_4 Z^{J-p-1}\rangle .
\end{aligned} \tag{60}$$

We should mention that the tree-level operator in the first line of (60) is normalized in such a way that its two point function is 1. Now we use the equation

$$\langle \mathcal{O}_{\beta,\psi\psi} | H_3 | \mathcal{O}_\alpha \rangle = E_{2,\alpha} \langle \mathcal{O}_{\beta,\psi\psi} | \mathcal{O}_{\alpha,\psi\psi} \rangle - \langle \mathcal{O}_{\beta,\psi\psi} | H_2 | \mathcal{O}_{\alpha,\psi\psi} \rangle\,, \tag{61}$$

being a direct consequence of the full eigenvalue equation for $|\hat{\mathcal{O}}_\alpha\rangle$ and the absence of $\mathcal{O}(g^3_{\rm YM})$ contributions to the scaling dimensions, to express the matrix element of H_3 between the leading bosonic term of the correlator and the subleading one involving fermions in terms of matrix elements of the 1-loop Hamiltonian H_2. In the sector we are interested in (e.g. scalar fermion pair of the type $\psi_1\psi_2$, $\bar{\psi}_3\bar{\psi}_4$ and the complex field Z) this can be written as

$$H_2 = \frac{\lambda}{2} \sum_{i=1}^{L} (1 - \varPi_{i,i+1}), \tag{62}$$

where $\varPi_{i,i+1}$ is the graded permutation operator.

Specialized to the case where $J = 2$, $n = 1$ we get

$$\langle \mathcal{O}^2_{1,\psi\psi} | \mathcal{O}^2_{1,\psi\psi} \rangle = 4 \times 2 \Big(g_{\rm YM} \frac{\sqrt{N}}{2\pi} \frac{1}{\sqrt{5}} \sin\frac{\pi}{5} \sin\frac{4\pi}{5} \Big)^2 \tag{63}$$

Here the factor of 4 appearing in (64) is due to the inner product of $|\psi^{1\alpha}\psi^2_\alpha Z\rangle - |\psi^{2\alpha}\psi^1_\alpha Z\rangle$, while the factor of 2 is due to the fact that there is a similar term with the fermions being $\bar{\psi}_3$ and $\bar{\psi}_4$. Finally the $\sin\frac{4\pi}{5}$ is due to the phase factor inside the sum of the second line of (60). Similarly, the matrix element of H_2 is given by

$$\langle \mathcal{O}^2_{1,\psi\psi}|H_2|\mathcal{O}^2_{1,\psi\psi}\rangle = 4\times 2\Big(g\frac{\sqrt{N}}{2\pi}\frac{1}{\sqrt{5}}\sin\frac{\pi}{5}\sin\frac{4\pi}{5}\Big)^2 3\lambda \tag{64}$$

Putting everything together we obtain for the two-loop energy of $\hat{O}^2_1$ the value

$$E_4 = \langle \mathcal{O}^2_1|H_4|\mathcal{O}^2_1\rangle + E_2\langle \mathcal{O}^2_{1,\psi\psi}|\mathcal{O}^2_{1,\psi\psi}\rangle = \frac{\lambda^2}{8}(-17+5\sqrt{5}). \tag{65}$$

This value is in complete agreement with the two-loop anomalous dimension of a level 4 descendant of the $\mathcal{O}^2_1$ primary state which belongs to an $SU(2)$ sub-sector. The precise form of this descendant operator is

$$\mathcal{O}_{desc.} = \frac{1}{J+1}\sum_{p=0}^{J}\cos\frac{\pi n(2p+1)}{J+1}\mathrm{Tr}\left(Z_1 Z^p Z_1 Z^{J-p}\right), \quad J=4,\ n=1, \tag{66}$$

and its two-loop anomalous dimension was found to be [9]

$$E_4 = 16\lambda^2\sin^4\frac{\pi n}{J+1}\Big(-\frac{1}{4}-\frac{\cos^2\frac{\pi n}{J+1}}{J+1}\Big), \tag{67}$$

which for $J=4$ and $n=1$ gives precisely (65).

We now proceed to the resolution of the mixing for the operators $\mathcal{O}^2_n$, $n=1,2$. The subleading terms of order g_{YM} are given by

$$|\mathcal{O}^2_{n,\psi\psi}\rangle = \sqrt{\lambda}\frac{1}{\sqrt{5}}\sin\frac{\pi n}{5}\sin\frac{4\pi n}{5}\big(|\psi^{1\alpha}\psi^2_\alpha Z\rangle - |\psi^{2\alpha}\psi^1_\alpha Z\rangle - |\bar{\psi}_{3\dot{\alpha}}\bar{\psi}_4^{\dot{\alpha}} Z\rangle + |\bar{\psi}_{4\dot{\alpha}}\bar{\psi}_3^{\dot{\alpha}} Z\rangle\big). \tag{68}$$

This gives

$$\langle \mathcal{O}^2_{2,\psi\psi}|\mathcal{O}^2_{1,\psi\psi}\rangle = -\frac{8}{5}\lambda\sin^2\frac{\pi}{5}\sin^2\frac{2\pi}{5}, \tag{69}$$

$$\langle \mathcal{O}^2_{2,\psi\psi}|H_2|\mathcal{O}^2_{1,\psi\psi}\rangle = -\frac{8}{5}\lambda\sin^2\frac{\pi}{5}\sin^2\frac{2\pi}{5}3\lambda \tag{70}$$

From (69) and (70) one obtains

$$\langle \mathcal{O}^2_{2,\psi\psi}|H_3|\mathcal{O}^2_1\rangle = -\frac{8}{5}\lambda^2\sin^2\frac{\pi}{5}\sin^2\frac{2\pi}{5}\Big(4\sin^2\frac{\pi}{5}-3\Big) \tag{71}$$

$$\langle \mathcal{O}^2_{1,\psi\psi}|H_3|\mathcal{O}^2_2\rangle = -\frac{8}{5}\lambda^2\sin^2\frac{\pi}{5}\sin^2\frac{2\pi}{5}\Big(4\sin^2\frac{2\pi}{5}-3\Big). \tag{72}$$

Finally, then the self-mixing contributions from the H_3 sector are given by

$$\hat{\mathcal{O}}_1^2 = \mathcal{O}_1^2 + \frac{\langle \mathcal{O}_2^2 | H_3 | \mathcal{O}_{1,\psi\psi}^2 \rangle}{4\lambda(\sin^2\frac{\pi}{5} - \sin^2\frac{2\pi}{5})}\,\mathcal{O}_2^2 = \mathcal{O}_1^2 + \lambda\frac{5-\sqrt{5}}{20}\mathcal{O}_2^2 = \hat{\mathcal{O}}_{4F}\,, \tag{73}$$

$$\hat{\mathcal{O}}_2^2 = \mathcal{O}_2^2 + \frac{\langle \mathcal{O}_1^2 | H_3 | \mathcal{O}_{2,\psi\psi}^2 \rangle}{4\lambda(\sin^2\frac{2\pi}{5} - \sin^2\frac{\pi}{5})}\,\mathcal{O}_1^2 = \mathcal{O}_2^2 + \lambda\frac{5+\sqrt{5}}{20}\mathcal{O}_1^2 = \hat{\mathcal{O}}_{4B}\,, \tag{74}$$

In a similar way, one can resolve the self-mixing for the operators $\mathcal{O}_n^3$, $n = 1, 2$.

$$\hat{\mathcal{O}}_1^3 = \mathcal{O}_1^3 + \lambda\frac{\sqrt{3}}{8}\mathcal{O}_2^3 = \hat{\mathcal{O}}_{5J}\,, \tag{75}$$

$$\hat{\mathcal{O}}_2^3 = \mathcal{O}_2^3 + \lambda\frac{\sqrt{3}}{8}\mathcal{O}_1^3 = \hat{\mathcal{O}}_{5E}\,, \tag{76}$$

Lastly, we focus on the length 4 primary operators $\mathcal{O}_{4A}$, $\mathcal{O}_{4E}$. In order to resolve the self-mixing here the most general form for the 1-loop Hamiltonian H_2 acting on one scalar and one fermion is needed. It is given by

$$\begin{aligned} H_2|\bar\psi_A \Phi^{BC}\rangle = {}& \frac{\lambda}{2}(|\bar\psi_A\Phi^{BC}\rangle - |\Phi^{BC}\bar\psi_A\rangle) + \frac{\lambda}{4}\big(\delta_A^B|\bar\psi_K\Phi^{KC}\rangle + \delta_A^B|\Phi^{KC}\bar\psi_K\rangle + \delta_A^C|\bar\psi_K\Phi^{BK}\rangle \\ & + \delta_A^C|\Phi^{BK}\bar\psi_K\rangle\big) + c\delta_A^B\sigma^\mu_{\alpha\dot\beta}\,|D_\mu\psi^C\rangle + c'\delta_A^C\sigma^\mu_{\alpha\dot\beta}\,|D_\mu\psi^B\rangle\,. \end{aligned} \tag{77}$$

The two last terms with the covariant derivatives acting on the fermion in the fundamental will not be needed in what follows. The action of H_2 on states where the scalar and the fermion are swaped or on states like $|\psi^A\Phi_{BC}\rangle$ should be obvious from (77). Let us write the two operators as

$$\begin{aligned} |\tilde{\mathcal{O}}_{4A}\rangle &= \tfrac{4}{\sqrt{738-102\sqrt{41}}}\big(|\Phi_{AB}\Phi^{AB}\Phi_{CD}\Phi^{CD}\rangle + \tfrac{5-\sqrt{41}}{4}|\Phi_{AB}\Phi_{CD}\Phi^{AB}\Phi^{CD}\rangle\big) + |\mathcal{O}_{4A,\psi\psi}\rangle \\ |\tilde{\mathcal{O}}_{4E}\rangle &= \tfrac{4}{\sqrt{738+102\sqrt{41}}}\big(|\Phi_{AB}\Phi^{AB}\Phi_{CD}\Phi^{CD}\rangle + \tfrac{5+\sqrt{41}}{4}|\Phi_{AB}\Phi_{CD}\Phi^{AB}\Phi^{CD}\rangle\big) + |\mathcal{O}_{4E,\psi\psi}\rangle\,, \end{aligned} \tag{78}$$

where

$$\begin{aligned} |\mathcal{O}_{4A,\psi\psi}\rangle &= -g_{\rm YM}\,c_A\,2\,(|\Phi_{AB}\psi^{A\alpha}\psi^B_\alpha\rangle - |\Phi^{AB}\bar\psi_{A\dot\alpha}\bar\psi^{\dot\alpha}_B\rangle)\frac{4}{\sqrt{738-102\sqrt{41}}} \\ |\mathcal{O}_{4E,\psi\psi}\rangle &= -g_{\rm YM}\,c_E\,2\,(|\Phi_{AB}\psi^{A\alpha}\psi^B_\alpha\rangle - |\Phi^{AB}\bar\psi_{A\dot\alpha}\bar\psi^{\dot\alpha}_B\rangle)\frac{4}{\sqrt{738+102\sqrt{41}}} \end{aligned} \tag{79}$$

The constants c_A and c_E are given by

$$c_{A/E} = -\frac{\sqrt{N}}{8\pi}(\frac{1}{2} - \frac{5\mp\sqrt{41}}{4})\,. \tag{80}$$

Furthermore, the normalizations appearing in (78) are such that the the leading scalar terms are normalized to 1. As above one can evaluate the following quantities

$$\langle \mathcal{O}_{4E,\psi\psi}|\mathcal{O}_{4A,\psi\psi}\rangle = 48 g_{\rm YM}^2 c_A c_E \frac{2}{3\sqrt{205}} \tag{81}$$

and

$$\langle\mathcal{O}_{4E,\psi\psi}|H_2|\mathcal{O}_{4A,\psi\psi}\rangle = 48g^2 c_A c_E \frac{2}{3\sqrt{205}}(3\lambda), \tag{82}$$

using (77). From this we deduce the matrix elements

$$\langle\mathcal{O}_{4E,\psi\psi}|H_3|\mathcal{O}_{4A}\rangle = 48g^2_{\rm YM} c_A c_E \frac{2}{3\sqrt{205}}(\frac{\lambda}{4}(13+\sqrt{41})-3\lambda)\,, \tag{83}$$

$$\langle\mathcal{O}_{4A,\psi\psi}|H_3|\mathcal{O}_{4E}\rangle = 48g^2_{\rm YM} c_A c_E \frac{2}{3\sqrt{205}}(\frac{\lambda}{4}(13-\sqrt{41})-3\lambda)\,. \tag{84}$$

Using these we may now also write down the self-mixing contributions originating from H_3

$$\hat{\mathcal{O}}_{4A} = \mathcal{O}_{4A} + 48g^2 c_A c_E \,\frac{2}{3\sqrt{205}}\frac{1-\sqrt{41}}{2\sqrt{41}}\mathcal{O}_{4E} = \mathcal{O}_{4A} + \lambda\frac{2(-1+\sqrt{41})}{41\sqrt{5}}\,\mathcal{O}_{4E}\,, \tag{85}$$

$$\hat{\mathcal{O}}_{4E} = \mathcal{O}_{4E} + 48g^2 c_A c_E \,\frac{2}{3\sqrt{205}}\frac{-1-\sqrt{41}}{2\sqrt{41}}\mathcal{O}_{4A} = \mathcal{O}_{4E} + \lambda\frac{2(1+\sqrt{41})}{41\sqrt{5}}\,\mathcal{O}_{4A}\,, \tag{86}$$

Finally, we may now state the complete self-mixing contributions to the operators that we are considering by combining the results of this and the previous subsection. We find

$$\hat{\mathcal{O}}^2_1 = \mathcal{O}^2_1 + \lambda\,\frac{1}{10}\frac{5+\sqrt{5}}{2}\,\mathcal{O}^2_2 = \mathcal{O}^2_1 + \frac{1}{10}\gamma_{22}\,\mathcal{O}^2_2 = \hat{\mathcal{O}}_{4F}\,, \tag{87}$$

$$\hat{\mathcal{O}}^2_2 = \mathcal{O}^2_2 + \lambda\,\frac{1}{10}\frac{5-\sqrt{5}}{2}\,\mathcal{O}^2_1 = \mathcal{O}^2_2 + \frac{1}{10}\gamma_{21}\,\mathcal{O}^2_1 = \hat{\mathcal{O}}_{4B}\,, \tag{88}$$

$$\hat{\mathcal{O}}^3_1 = \mathcal{O}^3_1 + \lambda\,\frac{3\sqrt{3}}{16}\,\mathcal{O}^3_2 = \mathcal{O}^3_1 + \frac{\sqrt{3}}{16}\,\gamma_{32}\mathcal{O}^3_2 = \hat{\mathcal{O}}_{5J}\,, \tag{89}$$

$$\hat{\mathcal{O}}^3_2 = \mathcal{O}^3_2 + \lambda\,\frac{\sqrt{3}}{16}\,\mathcal{O}^3_1 = \mathcal{O}^3_2 + \frac{\sqrt{3}}{16}\,\gamma_{31}\mathcal{O}^3_1 = \hat{\mathcal{O}}_{5E}\,, \tag{90}$$

$$\hat{\mathcal{O}}_{4A} = \mathcal{O}_{4A} + \lambda\,\frac{-7+2\sqrt{41}}{41\sqrt{5}}\,\mathcal{O}_{4E}\,, \tag{91}$$

$$\hat{\mathcal{O}}_{4E} = \mathcal{O}_{4E} + \lambda\,\frac{7+2\sqrt{41}}{41\sqrt{5}}\,\mathcal{O}_{4A}\,, \tag{92}$$

where $\gamma_{Jn} = 4\lambda\sin^2\frac{\pi n}{J+3}$ is the 1-loop anomalous dimension of the operator $\mathcal{O}^J_n$.

4.3 Fermionic and derivative mixing terms for the $\mathcal{O}_{4A}$ and $\mathcal{O}_{4E}$ operators

What remains to be found are the bi-fermionic and bi-derivative mixing contributions for the scalar $\mathcal{O}_{4A}$ and $\mathcal{O}_{4E}$ operators. Let us rewrite these operators in complex notation[5]:

$$\mathcal{O}^{(0)}_{4A/E} = {\rm Tr}\left(\Phi_{AB}\Phi^{AB}\Phi_{CD}\Phi^{CD}\right) + \alpha_{A/E}{\rm Tr}\left(\Phi_{AB}\Phi_{CD}\Phi^{AB}\Phi^{CD}\right) \tag{93}$$

[5]Our conventions are stated in appendix A.

where $\alpha_{A/E} = \frac{5\mp\sqrt{41}}{4}$.

At orders $g_{\rm YM}$ and $g^2_{\rm YM}$ we expect to find subleading mixing terms, singlets under the $SU(4)$ R-symmetry group and with naive scaling dimension four, containing respectively two fermions and two derivative impurities. The natural candidates are then:

$$c_1\, g_{\rm YM}\, {\rm Tr}\left(\Phi_{AB}\psi^A_\alpha\psi^B_\beta\right)\epsilon^{\alpha\beta} + c_2\, g_{\rm YM}\, {\rm Tr}\left(\Phi^{AB}\bar\psi^{\dot\alpha}_A\bar\psi^{\dot\beta}_B\right)\epsilon_{\dot\alpha\dot\beta} \tag{94}$$

and

$$d\, g^2_{\rm YM}\, D_\mu\Phi_{AB}D^\mu\Phi^{AB}. \tag{95}$$

We can compute the mixing coefficients by requiring that this operator - being a highest weight state - is annihilated by the action of the superconformal charges $\bar S^{\dot\alpha}_A$ and S^A_α for any $A = 1,\ldots,4$ at any order in perturbation theory. As it is composed by scalar fields only, the leading term is trivially annihilated by the superconformal charges at order $g^0_{\rm YM}$. This is no longer true at higher orders as the superconformal charges receive quantum corrections, thus their action on the leading term should be compensated by the action of S and $\bar S$ at lower orders on (94) and (95).

Let us first resolve the mixing at order $g_{\rm YM}$, demanding that the action S and $\bar S$ at this order on (93) is cancelled by the tree-level action of the same supercharges on the two terms in (94). These are determined by the contraction of the relevant term of the supercurrents in eqs (154).

Since these terms have opposite sign, we can write $c_1 = -c_2 = c$.

Let us focus on the variation generated by the charge $\bar S^{\dot\alpha}_{A=1}$ at order g_{YM}, and let us first compute the coefficient c by specializing the action of $\bar S^{\dot\alpha}_1$ on $\mathcal{O}^{(0)}_{4A/E}$ from (93). From equation (156), we get

$$\bar S^{\dot\alpha}_1\Phi_{AB}\Phi_{CD} = {\rm i}\frac{g_{\rm YM}N}{32\pi^2}\left(\epsilon_{1AB[C}\bar\psi^{\dot\alpha}_{D]} - \epsilon_{1CD[A}\bar\psi^{\dot\alpha}_{B]}\right), \tag{96}$$

where $\epsilon_{1AB[C}\bar\psi_{D]} = \frac{1}{2}(\epsilon_{1ABC}\bar\psi_D - \epsilon_{1ABD}\bar\psi_C)$.

We can act with $\bar S^{\dot\alpha}_1$ on ${\rm Tr}\left(\Phi_{AB}\Phi_{A'B'}\Phi_{CD}\Phi_{C'D'}\right)$. Then, we must contract the result with $\epsilon^{ABA'B'}\epsilon^{CDC'D'}$ and $\epsilon^{ABCD}\epsilon^{A'B'C'D'}$ to obtain the action of $\bar S$ at order-$g_{\rm YM}$ on the first and second terms of (93) respectively. We obtain

$$\bar S^{\dot\alpha}_1{\rm Tr}\left(\Phi_{AB}\Phi_{A'B'}\Phi_{CD}\Phi_{C'D'}\right)\epsilon^{ABA'B'}\epsilon^{CDC'D'} = -{\rm i}\frac{g_{\rm YM}N}{2\pi^2}{\rm Tr}\left(\Phi^{AB}[\bar\psi^{\dot\alpha}_B,\Phi_{1A}]\right) \tag{97}$$

$$\bar S^{\dot\alpha}_1{\rm Tr}\left(\Phi_{AB}\Phi_{A'B'}\Phi_{CD}\Phi_{C'D'}\right)\epsilon^{ABCD}\epsilon^{A'B'C'D'} = {\rm i}\frac{g_{\rm YM}N}{\pi^2}{\rm Tr}\left(\Phi^{AB}[\bar\psi^{\dot\alpha}_B,\Phi_{1A}]\right). \tag{98}$$

Thus

$$\bar S^{\dot\alpha}_1\mathcal{O}^{(0)}_{4A/E} = {\rm i}\frac{g_{\rm YM}N}{4\pi^2}\left(-\frac{1}{2}+\alpha_{A/E}\right){\rm Tr}\left(\Phi^{AB}[\bar\psi^{\dot\alpha}_B,\Phi_{1A}]\right). \tag{99}$$

Now let us consider the tree-level action of $\bar S_1$ on the subleading term in (94). Since (see eq. (155))

$$\bar S^{\dot\alpha}_1\bar\psi_{B\,\dot\beta} = 4\sqrt{2}\,{\rm i}\,\Phi_{1B}\delta^{\dot\alpha}_{\dot\beta}, \tag{100}$$

we have

$$\bar{S}_1^{\dot{\alpha}}\mathrm{Tr}\left(\Phi^{AB}\bar{\psi}_A^{\dot{\gamma}}\bar{\psi}_B^{\dot{\beta}}\right)\epsilon_{\dot{\gamma}\dot{\beta}} = 4\sqrt{2}\,\mathrm{i}\,\mathrm{Tr}\left(\Phi^{AB}[\bar{\psi}_B^{\dot{\alpha}},\Phi_{1A}]\right). \tag{101}$$

Recalling that $c_2 = -c_1 = -c$ one has

$$c = -\frac{N}{16\sqrt{2}\pi^2}\left(\frac{1}{2} - \alpha_{A/E}\right). \tag{102}$$

We now proceed to calculate the mixing of the operators 4A/4E with operators involving derivative terms. There is a single candidate consistent with the dimension of 4A/4E which is four and with the fact that the 4A/4E operators are singlets under both the $SU(4)$ and the Lorentz group. We denote this mixing term by $d\mathrm{Tr}\left(D_\mu\Phi_{AB}D^\mu\Phi^{AB}\right)$. An easy way to determine d is by demanding orthogonality of the 4A/4E operators with the Konishi up to order g^2. It is enough to consider just part of the Konishi operator, namely $\mathrm{Tr}\left(Z\bar{Z}\right)$. All other terms can be manipulated in a similar way and give the same result for d.

Firstly, we write the relevant Yukawa term as $-4\sqrt{2}g\,\mathrm{Tr}\left(\Phi_{AB}\psi^{\alpha B}\psi_\alpha^A\right)$. Then we focus on the 2-point function

$$\langle\mathrm{Tr}\left(\bar{\psi}_{\dot{\alpha}1}\bar{\psi}_2^{\dot{\alpha}}Z\right)(x)\;\;\mathrm{Tr}\left(Z\bar{Z}\right)(0)\rangle \tag{103}$$

Inserting the Yukawa and making the contractions we get

$$\begin{aligned}-2\sqrt{2}\,g\,i\,N^3\,\frac{1}{2^4}\,\Delta(x)\int d^4z\,\Delta(z)\,(i\,\sigma^\mu_{\alpha\dot{\alpha}}\partial^x_\mu\Delta(x-z))\,(-i\,\bar{\sigma}^{\nu\dot{\alpha}\alpha}\partial^z_\nu\Delta(x-z)) = \\ 2\sqrt{2}\,gN^3\,\frac{1}{2^4}\,\Delta^3(x),\end{aligned} \tag{104}$$

where to pass from the first to the second line of (104) we have used the identities

$$\sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{\nu\dot{\alpha}\alpha} = 2\eta_{\mu\nu} \tag{105}$$

$$\int d^4z\Delta(z)\partial^x_\mu\Delta(x-z)\partial^\mu_x\Delta(x-z) = -\frac{i}{2}\Delta^2(x). \tag{106}$$

Furthermore, the factor N^3 comes from the fact that the corresponding diagram has 3 loops while the factor of $1/2^4$ is related to the fact that there are 4 propagators each of which brings a factor of 1/2. To get the full contribution of the bi-fermion term for the operators 4A/4E we should multiply the result of (104) by 4. The first 2 is to take into account the contribution when the fermions $\bar{\psi}_1\bar{\psi}_2$ in the 4A/4E state are substituted by $\psi^3\psi^4$ while the second one to take into account the contributions when the Yukawa vertex is contracted with the $\bar{Z}$ field of the Konishi. Putting everything together and multiplying by the coefficient of the bi-fermion term which is $g\frac{N}{16\sqrt{2}\pi^2}\frac{3\mp\sqrt{41}}{4}$ for 4A and 4E respectively we get the final contribution from the bi-fermion term to be

$$\frac{g^2N^4}{16\,2\pi^2}\frac{3\mp\sqrt{41}}{4} \tag{107}$$

We now turn to the contribution of the bi-derivative insertions

$$d\,\langle\mathrm{Tr}\left(D^\mu\bar{Z}D_\mu Z\right)(x)\;\;\mathrm{Tr}\left(Z\bar{Z}\right)(0)\rangle, \tag{108}$$

where d is the coefficient which we need to determine. These involve only free contractions and read

$$d\,N^2\,\frac{1}{2^2}\partial^x_\mu\Delta(x)\partial^\mu_x\Delta(x)=d\,N^2\,\frac{1}{2^2}(-16\pi^2)\,\Delta^3(x) \tag{109}$$

Demanding that the sum of (107) and (109) is zero we deduce

$$d=\frac{g^2N^2}{2^7\pi^4}\frac{3\mp\sqrt{41}}{4}. \tag{110}$$

Finally, we should mention that the leading term of the 4A/4E operators and the Konishi have no overlap at order g^2.

5 Operators up to length $L=5$

In this short section we now collect the results of the previous chapter and write down the explicit form of the non-BPS operators we are going to use in the computation of the three point functions

- $L=2$

$$\mathcal{K}=\frac{8\pi^2}{\sqrt{3}N}\mathrm{Tr}\left(\Phi_{AB}\Phi^{AB}\right) \tag{111}$$

- $L=3$

$$\hat{\mathcal{O}}^1_1=\frac{8\pi^3}{N^{\frac{3}{2}}}\left[\mathrm{Tr}\left(\Phi^{AB}\Phi_{AB}Z\right)+\mathrm{Tr}\left(\Phi^{AB}Z\Phi_{AB}\right)\right] \tag{112}$$

- $L=4$

$$\begin{aligned}\hat{\mathcal{O}}^2_1=&\frac{(8\pi^2)^2}{N^2\sqrt{5}}\left(1+g^2_{\mathrm{YM}}\frac{N}{32\pi^2}(3-\sqrt{5})\right)\sum_{p=0}^{2}\cos\frac{\pi(2p+3)}{5}\mathrm{Tr}\left(\Phi^{AB}Z^p\Phi_{AB}Z^{2-p}\right)\\&+g_{\mathrm{YM}}\frac{\pi^2}{N}\sqrt{2}(\sqrt{5}-1)\Big[\mathrm{Tr}\left(\psi^{[1\,\alpha}\psi^{2]}_{\alpha}Z\right)-\mathrm{Tr}\left(\bar{\psi}_{[3\,\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}_{4]}Z\right)\Big]\\&+g^2_{\mathrm{YM}}\frac{\sqrt{5}-1}{16}\mathrm{Tr}\left(D_\mu ZD^\mu Z\right)\\&+g^2_{\mathrm{YM}}\frac{4\pi^2}{5N}(\sqrt{5}+1)\sum_{p=0}^{2}\cos\frac{2\pi(2p+3)}{5}\mathrm{Tr}\left(\Phi^{AB}Z^p\Phi_{AB}Z^{2-p}\right)\end{aligned} \tag{113}$$

$$\begin{aligned}\hat{\mathcal{O}}^2_2=&\frac{(8\pi^2)^2}{N^2\sqrt{5}}\left(1+g^2_{\mathrm{YM}}\frac{N}{32\pi^2}(3+\sqrt{5})\right)\sum_{p=0}^{2}\cos\frac{2\pi(2p+3)}{5}\mathrm{Tr}\left(\Phi^{AB}Z^p\Phi_{AB}Z^{2-p}\right)\\&-g_{\mathrm{YM}}\frac{\pi^2}{N}\sqrt{2}(1+\sqrt{5})\Big[\mathrm{Tr}\left(\psi^{[1\,\alpha}\psi^{2]}_{\alpha}Z\right)-\mathrm{Tr}\left(\bar{\psi}_{[3\,\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}_{4]}Z\right)\Big]\\&+g^2_{\mathrm{YM}}\frac{1+\sqrt{5}}{16}\mathrm{Tr}\left(D_\mu ZD^\mu Z\right)+\end{aligned} \tag{114}$$

$$+ g_{\rm YM}^2 \frac{4\pi^2}{5N}(\sqrt{5}-1)\sum_{p=0}^{2}\cos\frac{\pi(2p+3)}{5}\mathrm{Tr}\left(\Phi^{AB}Z^p\Phi_{AB}Z^{2-p}\right)$$

$$\hat{\mathcal{O}}_{4A} =\Big(\frac{8\pi^2}{N}\Big)^2\mathcal{N}_A\bigg\{\mathrm{Tr}\left(\Phi_{AB}\Phi^{AB}\Phi_{CD}\Phi^{CD}\right)+\frac{5-\sqrt{41}}{4}\mathrm{Tr}\left(\Phi_{AB}\Phi_{CD}\Phi^{AB}\Phi^{CD}\right) \tag{115}$$

$$- g_{\rm YM}\frac{N}{16\sqrt{2}\pi^2}\frac{3-\sqrt{41}}{4}\Big[\mathrm{Tr}\left(\Phi_{AB}\psi^{A\alpha}\psi_\alpha^B\right)-\mathrm{Tr}\left(\Phi^{AB}\bar{\psi}_{A\dot{\alpha}}\bar{\psi}_B^{\dot{\alpha}}\right)\Big]$$

$$+g_{\rm YM}^2\frac{N^2}{2^7\pi^4}\frac{3-\sqrt{41}}{4}\mathrm{Tr}\left(D_\mu\Phi_{AB}D^\mu\Phi^{AB}\right)\bigg\}$$

$$- g_{\rm YM}^2\frac{16\pi^2}{N}\frac{(7-2\sqrt{41})}{41\sqrt{5}}\mathcal{N}_E\Big[\mathrm{Tr}\left(\Phi_{AB}\Phi^{AB}\Phi_{CD}\Phi^{CD}\right)+\frac{5+\sqrt{41}}{4}\mathrm{Tr}\left(\Phi_{AB}\Phi_{CD}\Phi^{AB}\Phi^{CD}\right)\Big]$$

$$\hat{\mathcal{O}}_{4E} =\Big(\frac{8\pi^2}{N}\Big)^2\mathcal{N}_E\bigg\{\mathrm{Tr}\left(\Phi_{AB}\Phi^{AB}\Phi_{CD}\Phi^{CD}\right)+\frac{5+\sqrt{41}}{4}\mathrm{Tr}\left(\Phi_{AB}\Phi_{CD}\Phi^{AB}\Phi^{CD}\right) \tag{116}$$

$$- g_{\rm YM}\frac{N}{16\sqrt{2}\pi^2}\frac{3+\sqrt{41}}{4}\Big[\mathrm{Tr}\left(\Phi_{AB}\psi^{A\alpha}\psi_\alpha^B\right)-\mathrm{Tr}\left(\Phi^{AB}\bar{\psi}_{A\dot{\alpha}}\bar{\psi}_B^{\dot{\alpha}}\right)\Big]$$

$$+g_{\rm YM}^2\frac{N^2}{2^7\,\pi^4}\frac{3+\sqrt{41}}{4}\mathrm{Tr}\left(D_\mu\Phi_{AB}D^\mu\Phi^{AB}\right)\bigg\}$$

$$+ g_{\rm YM}^2\frac{16\pi^2}{N}\frac{(7+2\sqrt{41})}{41\sqrt{5}}\mathcal{N}_A\Big[\mathrm{Tr}\left(\Phi_{AB}\Phi^{AB}\Phi_{CD}\Phi^{CD}\right)+\frac{5-\sqrt{41}}{4}\mathrm{Tr}\left(\Phi_{AB}\Phi_{CD}\Phi^{AB}\Phi^{CD}\right)\Big]$$

where the antisymmetrization of the indices is defined as $\psi^{[1\,\alpha}\psi_\alpha^{2]}=\frac{1}{2}(\psi^{1\,\alpha}\psi_\alpha^2-\psi^{2\,\alpha}\psi_\alpha^1)$ and the normalization reads $\mathcal{N}_{4A/4E}=\frac{4}{\sqrt{738\mp102\sqrt{41}}}\Big(1+\frac{g_{\rm YM}^2N}{4\pi^2}\frac{25\mp3\sqrt{41}}{246\mp34\sqrt{41}}+\mathcal{O}(g_{\rm YM}^4)\Big)$.

- $L=5$

$$\hat{\mathcal{O}}_1^3 =\Big(\frac{8\pi^2}{N}\Big)^{\frac{5}{2}}\frac{1}{\sqrt{6}}\Big(1+g_{\rm YM}^2\frac{N}{32\pi^2}\Big)\sum_{p=0}^{3}\cos\frac{\pi(2p+3)}{6}\mathrm{Tr}\left(\Phi^{AB}Z^p\Phi_{AB}Z^{3-p}\right) \tag{117}$$

$$+ g_{\rm YM}\frac{4\sqrt{2}\pi^3}{\sqrt{3}N^{\frac{3}{2}}}\sum_{p=0}^{2}\sin\frac{\pi(2p+4)}{6}\Big[\mathrm{Tr}\left(\psi^{1\,\alpha}Z^p\psi_\alpha^2Z^{2-p}\right)-\mathrm{Tr}\left(\bar{\psi}_{3\,\dot{\alpha}}Z^p\bar{\psi}_4^{\dot{\alpha}}Z^{2-p}\right)\Big]$$

$$- g_{\rm YM}^2\frac{\pi}{8\sqrt{N}}\left[\mathrm{Tr}\left(D_\mu ZD^\mu ZZ\right)+\mathrm{Tr}\left(D_\mu ZZD^\mu Z\right)\right] \tag{118}$$

$$+ g_{\rm YM}^2\frac{6\pi^3}{N^{\frac{3}{2}}}\sum_{p=0}^{3}\cos\frac{\pi(2p+3)}{3}\mathrm{Tr}\left(\Phi^{AB}Z^p\Phi_{AB}Z^{3-p}\right)$$

$$\hat{\mathcal{O}}_2^3 =\Big(\frac{8\pi^2}{N}\Big)^{\frac{5}{2}}\frac{1}{\sqrt{6}}\Big(1+g_{\rm YM}^2\frac{3N}{32\pi^2}\Big)\sum_{p=0}^{3}\cos\frac{\pi(2p+3)}{3}\mathrm{Tr}\left(\Phi^{AB}Z^p\Phi_{AB}Z^{3-p}\right) \tag{119}$$

$$
\begin{aligned}
&+ g_{\rm YM}\frac{4\sqrt{2}\pi^3}{N^{\frac{3}{2}}}\sum_{p=0}^{2}\sin\frac{\pi(2p+4)}{3}\Big[\mathrm{Tr}\left(\psi^{1\,\alpha}Z^p\psi^2_\alpha Z^{2-p}\right)-\mathrm{Tr}\left(\bar\psi_{3\,\dot\alpha}Z^p\bar\psi_4^{\dot\alpha}Z^{2-p}\right)\Big]\\
&+ g^2_{\rm YM}\frac{\sqrt{3}\pi}{8\sqrt{N}}\big[\mathrm{Tr}\left(D_\mu ZD^\mu ZZ\right)+\mathrm{Tr}\left(D_\mu ZZD^\mu Z\right)\big] \qquad (120)\\
&+ g^2_{\rm YM}\frac{2\pi^3}{N^{\frac{3}{2}}}\sum_{p=0}^{3}\cos\frac{\pi(2p+3)}{6}\mathrm{Tr}\left(\Phi^{AB}Z^p\Phi_{AB}Z^{3-p}\right)
\end{aligned}
$$

- **BPS Operators**

  We will also need explicit forms of the protected 1/2 BPS operators beyond the lengths three and four. The maximally charged operators are of course simply $\mathrm{Tr}\,(Z)^J$ carrying $U(1)$ charge J. At length three we note the BPS operator

$$
\begin{aligned}
\mathcal{O}_{3C,ijk} = \mathrm{Tr}\left(Z_i\bar Z_jZ_k+\bar Z_jZ_iZ_k\right)-\frac{1}{4}\,\delta_{ij}\mathrm{Tr}\left(Z_p\bar Z_pZ_k+\bar Z_pZ_pZ_k\right)\\
-\frac{1}{4}\,\delta_{jk}\mathrm{Tr}\left(Z_p\bar Z_pZ_i+\bar Z_pZ_pZ_i\right),\ \ i,j,k=1,2,3 \qquad (121)
\end{aligned}
$$

  At length four we have the neutral BPS operators:

$$
\begin{aligned}
\mathcal{O}_{4G,neutral} = 2\Big[&\mathrm{Tr}\left(4Z_2Z_2\bar Z_2\bar Z_2+2Z_2\bar Z_2Z_2\bar Z_2\right)-\frac{4}{5}\mathrm{Tr}\left(Z_{(p}Z_2\bar Z_{p)}\bar Z_2+\bar Z_{(p}Z_{p)}Z_{(2}\bar Z_{2)}\right)\Big]\\
-\Big[&\mathrm{Tr}\left(4Z_1Z_1\bar Z_1\bar Z_1+2Z_1\bar Z_1Z_1\bar Z_1\right)-\frac{4}{5}\mathrm{Tr}\left(Z_{(p}Z_1\bar Z_{p)}\bar Z_1+\bar Z_{(p}Z_{p)}Z_{(1}\bar Z_{1)}\right)\Big]\\
-\Big[&\mathrm{Tr}\left(4Z_3Z_3\bar Z_3\bar Z_3+2Z_3\bar Z_3Z_3\bar Z_3\right)-\frac{4}{5}\mathrm{Tr}\left(Z_{(p}Z_3\bar Z_{p)}\bar Z_3+\bar Z_{(p}Z_{p)}Z_{(3}\bar Z_{3)}\right)\Big], \qquad (122)
\end{aligned}
$$

  The charge two BPS operator reads

$$
\begin{aligned}
\mathcal{O}_{4G,ijkl} = &\mathrm{Tr}\left(Z_iZ_jZ_k\bar Z_l+Z_iZ_j\bar Z_lZ_k+Z_iZ_kZ_j\bar Z_l+Z_iZ_k\bar Z_lZ_j+Z_i\bar Z_lZ_jZ_k+Z_i\bar Z_lZ_kZ_j\right)\\
&-\frac{1}{5}\,\delta_{kl}\mathrm{Tr}\left(Z_iZ_{(p}Z_j\bar Z_{p)}+Z_{(i}Z_{j)}Z_{(p}\bar Z_{p)}\right)-\frac{1}{5}\,\delta_{jl}\mathrm{Tr}\left(Z_kZ_{(p}Z_i\bar Z_{p)}+Z_{(k}Z_{i)}Z_{(p}\bar Z_{p)}\right)\\
&-\frac{1}{5}\,\delta_{il}\mathrm{Tr}\left(Z_kZ_{(p}Z_j\bar Z_{p)}+Z_{(j}Z_{k)}Z_{(p}\bar Z_{p)}\right),\ \ i,j,k,l=1,2,3 \qquad (123)
\end{aligned}
$$

  where the repeated indices $p,q=1,2,3$ are summed over and the bracket in the indices means symmetrisation. Namely,

$$
\mathrm{Tr}\left(Z_{(p}Z_j\bar Z_{p)}\bar Z_l\right)=\sum_{p=1}^{3}\mathrm{Tr}\left(Z_pZ_j\bar Z_p\bar Z_l+\bar Z_pZ_jZ_p\bar Z_l\right) \qquad (124)
$$

$$
\mathrm{Tr}\left(\bar Z_{(p}Z_{p)}Z_{(j}\bar Z_{l)}\right)=\sum_{p=1}^{3}\mathrm{Tr}\left(Z_p\bar Z_pZ_{(j}\bar Z_{l)}+\bar Z_pZ_pZ_{(j}\bar Z_{l)}\right)=
$$

$$
\sum_{p=1}^{3} \mathrm{Tr}\left(Z_p \bar{Z}_p Z_j \bar{Z}_l + Z_p \bar{Z}_p \bar{Z}_l Z_j + \bar{Z}_p Z_p Z_j \bar{Z}_l + \bar{Z}_p Z_p \bar{Z}_l Z_j\right). \tag{125}
$$

Length	Class	$SU(4)^{\text{parity}}_{\text{length}}$ Rep.	Dim.	$8\pi^2\,\gamma$	Operator	Mixing
2	2A	$[0,0,0]_2^+$	1	6	$\mathcal{K}$	no mixing
	2B	$[0,2,0]_2^+$	20	0	CPO	no mixing
3	3B	$[0,1,0]_3^-$	6	4	$\mathcal{O}^{J=1}_{n=1}$	resolved
	3C	$[0,3,0]_3^-$	50	0	CPO	no mixing
4	4A	$[0,0,0]_4^+$	1	$\frac{1}{2}\,(13+\sqrt{41})$	$\mathcal{O}_{4A}$	resolved
	4E	$[0,0,0]_4^+$	1	$\frac{1}{2}\,(13-\sqrt{41})$	$\mathcal{O}_{4E}$	resolved
	4B	$[0,2,0]_4^+$	20	$5+\sqrt{5}$	$\mathcal{O}^{J=2}_{n=2}$	resolved
	4F	$[0,2,0]_4^+$	20	$5-\sqrt{5}$	$\mathcal{O}^{J=2}_{n=1}$	resolved
	4C	$[2,0,2]_4+[1,0,1]_4^-$	84 + 15	6		
	4G	$[0,4,0]_4^+$	105	0	CPO	no mixing
5	**5A**	$[0,0,2]_5^+ + [2,0,0]_5^+$	10 + $\bar{1}\bar{0}$	$7+\sqrt{13}$		
	5H	$[0,0,2]_5^+ + [2,0,0]_5^+$	10 + $\bar{1}\bar{0}$	$7-\sqrt{13}$		
	5D	$[0,1,0]_5^-$ + desc	6 + 252	$5+\sqrt{5}$		
	5I	$[0,1,0]_5^-$ + desc	6 + 252	$5-\sqrt{5}$		
	5F	$[1,1,1]_5^+ + [1,1,1]_5^-$	64 + 64	5		
	5J	$[0,3,0]_5^-$	50	2	$\mathcal{O}^{J=3}_{n=1}$	resolved
	5E	$[0,3,0]_5^-$ + desc	50 + 140	6	$\mathcal{O}^{J=3}_{n=2}$	resolved
	5K	$[0,5,0]_5^-$	196	0	CPO	no mixing
	5B	$[0,1,0]_5^-$	**6+6**	10		

Table 1: List of all scalar conformal primary operator up to length 5 with their one-loop anomalous dimensions. Degenerate classes of operators are printed in bold-face. $\mathcal{K}$ denotes the Konishi and CPO chiral primary operators. The $\mathcal{O}^J_n$ refer to the BMN singlet operators in the nomenclature of (44). In the last column the resolved mixing problem with fermion, derivative and self-mixings of section 4 are displayed.

6 Results

The final result for the structure constants arises from two contributions: The radiative one-loop corrections discussed in section 3 as well as the corrections arising from the operator mixing effects

spelled out in section 5, which in principle enable one to straightforwardly compute three-point functions involving scalar operators up to length five by combinatorial means.

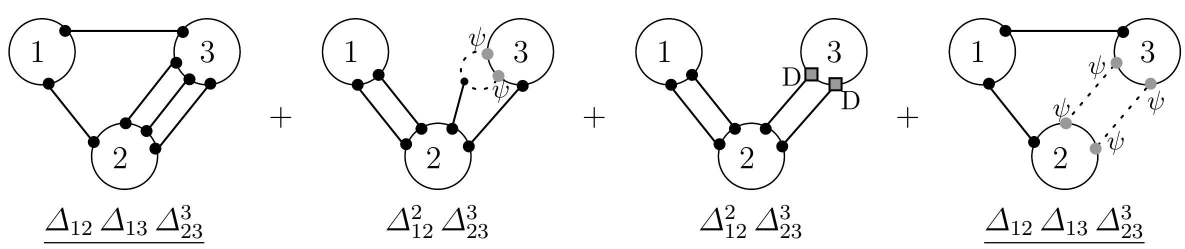
Let us begin with the dressing formulae to find the radiative corrections. Clearly, due to the need to sum over all permutations in these dressing formulae the complexity in the computations grows fast and needs to be done on a computer. This has been implemented in a two step procedure. Starting with an arbitrarily chosen basis of operators all two-point functions are computed and then diagonalized. All scalar operators up to length five are detailed in figure 1. Similarly all three-point functions are computed in the original basis and then projected to the diagonal basis where the structure constants can be extracted. For operators up to length three this was done algebraically with a MATHEMATICA program. Starting with length four the mixing matrix diagonalization could not be performed algebraically any longer and we had to resort to numerics using Matlab. Once the diagonal basis was constructed the numerically obtained structure constants could in most cases be again fitted to algebraic expressions derived by the algebraic form of the one-loop scaling dimensions.

Secondly the contribution from the structure constants from the mixing terms of section 5 were found as well. These arise from tree-level contractions involving the operator corrections due to double bi-fermion, bi-derivative and self-mixing, as well as bi-fermion corrections of one operator and a Yukawa-interaction. It turns out that the relevant contibutions securing conformal symmetry arise from suitable tree-level correlators only - the Yukawa contributions always cancel.

Below we list our main results sorted by correlator classes which are listed in the tables 2, 3, 4 and 5[6]. Note that only three-point functions which do not vanish at tree-level are listed. We also stress that the majority of results for the radiative corrections to the fractions $C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}|_{\text{loop}}$ have been obtained numerically and the quoted analytical results represents a biases fit allowing as non-rational factors only the square root term appearing in the anomalous scaling dimensions of the operators involved in the patricular three-point function. The numerical precision in theses fits is typically of order 10^{-5} or better, for the raw data see the appendix A.2 of [51]. Finally, the analytically obtained results are highlighted in bold-face letters.

6.1 $\langle 2|4|4\rangle$ correlators

Here a diagramatic analysis reveals that only the double bi-fermionic mixing and the scalar self-mixings will contribute to the three-point correlator, whereas the Yukawa-vertex insertion cancels against the bi-derivative mixing contributions. This follows by considering the propagator dependances of these terms.



[6]Here we have used everywhere $\lambda = g^2_{YM}N$

The underlined terms contribute.

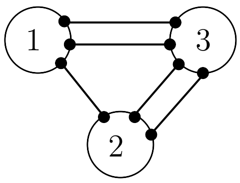
We hence only need to determine the ratio

$$\frac{C^{(1)}_{\alpha\beta\gamma}}{C^{(0)}_{\alpha\beta\gamma}}\bigg|_{\text{mixing}} = \frac{\langle 2|4_{\psi\psi}|4\rangle + \langle 2|4|4_{\psi\psi}\rangle + \langle 2|4_{\text{self}}|4\rangle + \langle 2|4|4_{\text{self}}\rangle}{\langle 2|4|4\rangle}\,. \tag{126}$$

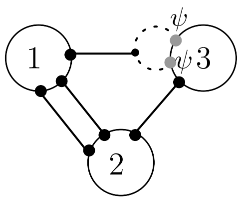
The highly involved evaluation of these correlators was performed with the help of a Mathematica program. In addition one has the radiative corrections in the pure $SO(6)$ sector whose form follows from the dressing procedure. We state the radiative contributions and the mixing contributions seperately and give the complete result in the final column, see table 2.

6.2 $\langle 3|3|4\rangle$ correlators

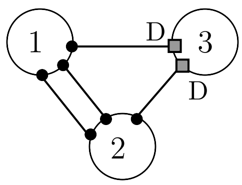
For the $\langle 3|3|4\rangle$ the diagramatic analysis of



$+$



$+$



$\underline{\Delta_{12}\,\Delta_{23}^2\,\Delta_{13}^2}$ $\qquad$ $\Delta_{12}^2\,\Delta_{13}^2\,\Delta_{23} + 1\leftrightarrow 2$ $\qquad$ $\Delta_{12}^2\,\left(\Delta_{13}^2\,\Delta_{23} + \Delta_{13}\,\Delta_{23}^2 - \underline{\Delta_{13}^2\,\Delta_{23}^2\,\Delta_{12}^{-1}}\right)$

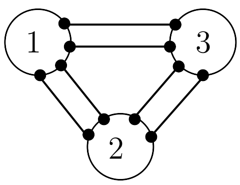
reveals that in the mixing sector we can only have corrections due to self-mixing, bi-fermion and bi-derivative insertions for the operator of engineering length four. Again, the Yukawa insertion cancels against part of the bi-derivative term (see appendix C), and the surviving terms arise just from the contributions

$$\frac{C^{(1)}_{\alpha\beta\gamma}}{C^{(0)}_{\alpha\beta\gamma}}\bigg|_{\text{mixing}} = \frac{\langle 3|3|4_{\text{self}}\rangle + \langle 3|4|4_{DD}\rangle}{\langle 3|4|4\rangle}\,. \tag{127}$$

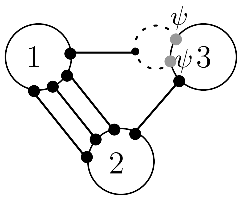
The results are exposed in table 3.

6.3 $\langle 4|4|4\rangle$ correlators

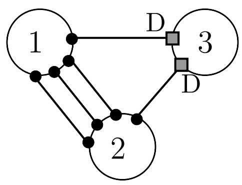
Turning to the $\langle 4|4|4\rangle$ correlators a similar diagramatic analysis tells us that now the Yukawa insertion cancels part of the double derivative corrections to the length four operators, while the remaining term yields the surviving contribution respecting conformal symmetry.



$+$



$+$



$\underline{\Delta_{12}^2\,\Delta_{23}^2\,\Delta_{13}^2}$ $\qquad$ $\Delta_{12}^3\,\Delta_{13}^2\,\Delta_{23} + 1\leftrightarrow 2$ $\qquad$ $\Delta_{12}^3\,\left(\Delta_{13}^2\,\Delta_{23} + \Delta_{13}\,\Delta_{23}^2 - \underline{\Delta_{13}^2\,\Delta_{23}^2\,\Delta_{12}^{-1}}\right)$

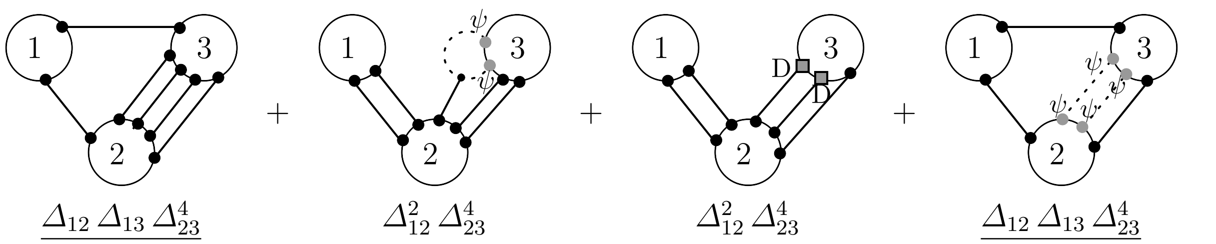
Hence, one here needs to evaluate the mixing contributions

$$\frac{C^{(1)}_{\alpha\beta\gamma}}{C^{(0)}_{\alpha\beta\gamma}}\bigg|_{\text{mixing}} = \frac{\langle 4|4|4_{DD}\rangle + \langle 4|4_{DD}|4\rangle + \langle 4_{DD}|4|4\rangle + \langle 4|4|4_{\text{self}}\rangle + \langle 4|4_{\text{self}}|4\rangle + \langle 4_{\text{self}}|4|4\rangle}{\langle 4|4|4\rangle}\,, \tag{128}$$

which are summarized together with the radiative corrections in table 4.

6.4 $\langle 2|5|5\rangle$ correlators

Finally the structure constants involving two length five operators and one length two operator are similarly controlled by the bi-fermi and self-mixing insertions, the Yukawa contribution cancels against the bi-derivative correction.



Hence, we evaluate the contributions

$$\frac{C^{(1)}_{\alpha\beta\gamma}}{C^{(0)}_{\alpha\beta\gamma}}\bigg|_{\text{mixing}} = \frac{\langle 2|5_{\psi\psi}|5_{\psi\psi}\rangle + \langle 2|5|5_{\text{self}}\rangle + \langle 2|5_{\text{self}}|5\rangle}{\langle 2|5|5\rangle}\,, \tag{129}$$

for the two cases in table 5.

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	$8\pi^2\gamma_\alpha$	$8\pi^2\gamma_\beta$	$8\pi^2\gamma_\gamma$	$\frac{-16\pi^2 C^{(1)}_{\alpha\beta\gamma}}{C^{(0)}_{\alpha\beta\gamma}}\vert_{\text{loop}}$	$\frac{-16\pi^2 C^{(1)}_{\alpha\beta\gamma}}{C^{(0)}_{\alpha\beta\gamma}}\vert_{\text{mixing}}$	Sum
2A	4A	4A	6	$\frac{13+\sqrt{41}}{2}$	$\frac{13+\sqrt{41}}{2}$	$\frac{1}{2}(25+\sqrt{41})$	$-\frac{3}{2}-\frac{7}{2\sqrt{41}}$	$11+\frac{17}{\sqrt{41}}$
2A	4B	4B	6	$5+\sqrt{5}$	$5+\sqrt{5}$	$11+\sqrt{5}$	$-\frac{3}{4}(3+\sqrt{5})$	$\frac{1}{4}(35+\sqrt{5})$
2A	4E	4E	6	$\frac{13-\sqrt{41}}{2}$	$\frac{13-\sqrt{41}}{2}$	$\frac{1}{2}(25-\sqrt{41})$	$-\frac{3}{2}+\frac{7}{2\sqrt{41}}$	$11-\frac{17}{\sqrt{41}}$
2A	4F	4F	6	$5-\sqrt{5}$	$5-\sqrt{5}$	$11-\sqrt{5}$	$-\frac{3}{4}(3-\sqrt{5})$	$\frac{1}{4}(35-\sqrt{5})$
2B	4A	4B	0	$\frac{13+\sqrt{41}}{2}$	$5+\sqrt{5}$	$5+\sqrt{5}$	$\frac{-3699+533\sqrt{5}-651\sqrt{41}-75\sqrt{205}}{1640}$	$\frac{4501+2173\sqrt{5}-651\sqrt{41}-75\sqrt{205}}{1640}$
2B	4A	4F	0	$\frac{13+\sqrt{41}}{2}$	$5-\sqrt{5}$	$5-\sqrt{5}$	$\frac{-3699-533\sqrt{5}-651\sqrt{41}+75\sqrt{205}}{1640}$	$\frac{4501-2173\sqrt{5}-651\sqrt{41}+75\sqrt{205}}{1640}$
2B	4B	4B	0	$5+\sqrt{5}$	$5+\sqrt{5}$	$\frac{2}{79}(115+14\sqrt{5})$	$-\frac{3}{395}\left(175+11\sqrt{5}\right)$	$\frac{1}{395}\left(615+107\sqrt{5}\right)$
2B	4B	4E	0	$5+\sqrt{5}$	$\frac{13-\sqrt{41}}{2}$	$5+\sqrt{5}$	$\frac{-3699+533\sqrt{5}+651\sqrt{41}+75\sqrt{205}}{1640}$	$\frac{4501+2173\sqrt{5}+651\sqrt{41}+75\sqrt{205}}{1640}$
2B	4B	4F	0	$5+\sqrt{5}$	$5-\sqrt{5}$	0	$-\frac{35}{3}$	$-\frac{35}{3}$
2B	4B	4G	0	$5+\sqrt{5}$	0	$5+\sqrt{5}$	$\frac{1}{10}\left(-25-7\sqrt{5}\right)$	$\frac{5}{2}+\frac{3}{2\sqrt{5}}$
2B	4E	4F	0	$\frac{13-\sqrt{41}}{2}$	$5-\sqrt{5}$	$5-\sqrt{5}$	$\frac{-3699-533\sqrt{5}+651\sqrt{41}-75\sqrt{205}}{1640}$	$\frac{4501-2173\sqrt{5}+651\sqrt{41}-75\sqrt{205}}{1640}$
2B	4F	4F	0	$5-\sqrt{5}$	$5-\sqrt{5}$	$\frac{2}{79}(115-14\sqrt{5})$	$\frac{3}{395}\left(11\sqrt{5}-175\right)$	$\frac{1}{395}\left(615-107\sqrt{5}\right)$
2B	4F	4G	0	$5-\sqrt{5}$	0	$5-\sqrt{5}$	$\frac{1}{10}\left(7\sqrt{5}-25\right)$	$\frac{5}{2}-\frac{3}{2\sqrt{5}}$

Table 2: The evaluated $\langle 2|4|4\rangle$ three-point correlators.

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	$8\pi^2\gamma_\alpha$	$8\pi^2\gamma_\beta$	$8\pi^2\gamma_\gamma$	$\frac{-16\pi^2 C^{(1)}_{\alpha\beta\gamma}}{C^{(0)}_{\alpha\beta\gamma}}\vert_{\text{loop}}$	$\frac{-16\pi^2 C^{(1)}_{\alpha\beta\gamma}}{C^{(0)}_{\alpha\beta\gamma}}\vert_{\text{mixing}}$	Sum
3B	3B	4A	4	4	$\frac{13+\sqrt{41}}{2}$	$\frac{1}{50}(261+9\sqrt{41})$	$\frac{2}{205}\left(21\sqrt{41}-121\right)$	$\frac{8281+789\sqrt{41}}{2050}$
3B	3B	4E	4	4	$\frac{13-\sqrt{41}}{2}$	$\frac{1}{50}(261-9\sqrt{41})$	$-\frac{2}{205}\left(121+21\sqrt{41}\right)$	$\frac{8281-789\sqrt{41}}{2050}$
3B	3B	4B	4	4	$5+\sqrt{5}$	$\frac{1}{11}(87+3\sqrt{5})$	$-\frac{1}{110}\left(175+3\sqrt{5}\right)$	$\frac{1}{110}\left(695+27\sqrt{5}\right)$
3B	3B	4F	4	4	$5-\sqrt{5}$	$\frac{1}{11}(87-3\sqrt{5})$	$\frac{1}{110}\left(123\sqrt{5}-355\right)$	$\frac{1}{110}\left(515+93\sqrt{5}\right)$

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	$8\pi^2\gamma_\alpha$	$8\pi^2\gamma_\beta$	$8\pi^2\gamma_\gamma$	$\frac{-16\pi^2 C^{(1)}_{\alpha\beta\gamma}}{C^{(0)}_{\alpha\beta\gamma}}\Big\vert_{\text{loop}}$	$\frac{-16\pi^2 C^{(1)}_{\alpha\beta\gamma}}{C^{(0)}_{\alpha\beta\gamma}}\Big\vert_{\text{mixing}}$	Sum
3B	3C	4B	4	0	$5+\sqrt{5}$	$\frac{1}{11}(39+7\sqrt{5})$	$-\frac{1}{110}\left(215+7\sqrt{5}\right)$	$\frac{7}{110}\left(25+9\sqrt{5}\right)$
3B	3C	4F	4	0	$5-\sqrt{5}$	$\frac{1}{11}(39-7\sqrt{5})$	$\frac{3}{110}\left(29\sqrt{5}-105\right)$	$\frac{1}{110}\left(75+17\sqrt{5}\right)$
3C	3C	4A	0	0	$\frac{13+\sqrt{41}}{2}$	$\frac{1}{2}(13+\sqrt{41})$	$\frac{2}{205}\left(371+89\sqrt{41}\right)$	$\frac{1}{210}\left(2849+461\sqrt{41}\right)$
3C	3C	4E	0	0	$\frac{13-\sqrt{41}}{2}$	$\frac{1}{2}(13-\sqrt{41})$	$-\frac{2}{205}\left(89\sqrt{41}-371\right)$	$\frac{1}{210}\left(2849-461\sqrt{41}\right)$
3C	3C	4B	0	0	$5+\sqrt{5}$	$5+\sqrt{5}$	$-\frac{1}{10}\left(45+17\sqrt{5}\right)$	$\frac{1}{10}\left(5-7\sqrt{5}\right)$
3C	3C	4F	0	0	$5-\sqrt{5}$	$5-\sqrt{5}$	$\frac{1}{10}\left(55-23\sqrt{5}\right)$	$\frac{1}{10}\left(105-33\sqrt{5}\right)$
3B	3C	4G	4	0	0	4	0	4

Table 3: The evaluated $\langle 3|3|4\rangle$ three-point correlators.

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	$8\pi^2\gamma_\alpha$	$8\pi^2\gamma_\beta$	$8\pi^2\gamma_\gamma$	$\frac{-16\pi^2 C^{(1)}_{\alpha\beta\gamma}}{C^{(0)}_{\alpha\beta\gamma}}\Big\vert_{\text{loop}}$	$\frac{-16\pi^2 C^{(1)}_{\alpha\beta\gamma}}{C^{(0)}_{\alpha\beta\gamma}}\Big\vert_{\text{mixing}}$	Sum
4A	4A	4A	$\frac{13+\sqrt{41}}{2}$	$\frac{13+\sqrt{41}}{2}$	$\frac{13+\sqrt{41}}{2}$	$\frac{1}{733}\left(7185+309\sqrt{41}\right)$	$\frac{6\left(11417\sqrt{41}-105667\right)}{150265}$	$\frac{3\left(279641+43949\sqrt{41}\right)}{150265}$
4A	4A	4E	$\frac{13+\sqrt{41}}{2}$	$\frac{13+\sqrt{41}}{2}$	$\frac{13-\sqrt{41}}{2}$	$\frac{1}{10}\left(21-\sqrt{41}\right)$	$-\frac{1}{820}\left(3847+383\sqrt{41}\right)$	$-\frac{1}{164}\left(425+93\sqrt{41}\right)$
4A	4B	4B	$\frac{13+\sqrt{41}}{2}$	$5+\sqrt{5}$	$5+\sqrt{5}$	12.3279656	$\frac{\sqrt{41}\left(1263+527\sqrt{5}\right)-88\left(108+41\sqrt{5}\right)}{410}$	7.59846
4A	4B	4F	$\frac{13+\sqrt{41}}{2}$	$5+\sqrt{5}$	$5-\sqrt{5}$	$\frac{1}{2}\left(9+\sqrt{41}\right)$	$\frac{-3149+205\sqrt{5}-171\sqrt{41}}{410}$	$\frac{-1304+205\sqrt{5}+34\sqrt{41}}{410}$
4A	4E	4E	$\frac{13+\sqrt{41}}{2}$	$\frac{13-\sqrt{41}}{2}$	$\frac{13-\sqrt{41}}{2}$	$\frac{1}{10}\left(21+\sqrt{41}\right)$	$\frac{1}{820}\left(383\sqrt{41}-3847\right)$	$\frac{1}{820}\left(301\sqrt{41}-2781\right)$
4A	4F	4F	$\frac{13+\sqrt{41}}{2}$	$5-\sqrt{5}$	$5-\sqrt{5}$	4.865786	$\frac{-8479+3280\sqrt{5}+1058\sqrt{41}-445\sqrt{205}}{410}$	3.05695
4A	4G	4G	$\frac{13+\sqrt{41}}{2}$	0	0	$\frac{1}{2}\left(13+\sqrt{41}\right)$	$\frac{1}{410}\left(459+151\sqrt{41}\right)$	$\frac{2}{205}\left(781+89\sqrt{41}\right)$
4B	4B	4E	$5+\sqrt{5}$	$5+\sqrt{5}$	$\frac{13-\sqrt{41}}{2}$	38.020253	$-\frac{4752}{205}-\frac{44}{5}\sqrt{5}-\frac{1263}{410}\sqrt{41}-\frac{527}{410}\sqrt{205}$	-42.9660
4B	4B	4G	$5+\sqrt{5}$	$5+\sqrt{5}$	0	$\frac{4}{19}\left(25+7\sqrt{5}\right)$	$\frac{1}{190}\left(-785-121\sqrt{5}\right)$	$\frac{1}{190}\left(215+159\sqrt{5}\right)$
4B	4E	4F	$5+\sqrt{5}$	$\frac{13-\sqrt{41}}{2}$	$5-\sqrt{5}$	$\frac{1}{2}\left(9-\sqrt{41}\right)$	$-\frac{3149}{410}+\frac{1}{2}\sqrt{5}+\frac{171}{410}\sqrt{41}$	$-\frac{652}{205}+\frac{1}{2}\sqrt{5}-\frac{17}{205}\sqrt{41}$

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	$8\pi^2\gamma_\alpha$	$8\pi^2\gamma_\beta$	$8\pi^2\gamma_\gamma$	$\frac{-16\pi^2 C^{(1)}_{\alpha\beta\gamma}}{C^{(0)}_{\alpha\beta\gamma}}\|_{\text{loop}}$	$\frac{-16\pi^2 C^{(1)}_{\alpha\beta\gamma}}{C^{(0)}_{\alpha\beta\gamma}}\|_{\text{mixing}}$	Sum
4E	4E	4E	$\frac{13-\sqrt{41}}{2}$	$\frac{13-\sqrt{41}}{2}$	$\frac{13-\sqrt{41}}{2}$	$\frac{1}{733}\left(7185-309\sqrt{41}\right)$	$-\frac{6\left(105667+11417\sqrt{41}\right)}{150265}$	$-\frac{3\left(43949\sqrt{41}-279641\right)}{150265}$
4E	4F	4F	$\frac{13-\sqrt{41}}{2}$	$5-\sqrt{5}$	$5-\sqrt{5}$	4.785995	$-\frac{8479}{410}+\frac{89\sqrt{\frac{5}{41}}}{2}+8\sqrt{5}-\frac{529}{5\sqrt{41}}$	1.01094
4E	4G	4G	$\frac{13-\sqrt{41}}{2}$	0	0	$\frac{1}{2}\left(13-\sqrt{41}\right)$	$\frac{1}{410}\left(459-151\sqrt{41}\right)$	$-\frac{2}{205}\left(89\sqrt{41}-781\right)$

Table 4: The evaluated $\langle 4\|4\|4\rangle$ three-point correlators.

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	$8\pi^2\gamma_\alpha$	$8\pi^2\gamma_\beta$	$8\pi^2\gamma_\gamma$	$\frac{-16\pi^2 C^{(1)}_{\alpha\beta\gamma}}{C^{(0)}_{\alpha\beta\gamma}}\|_{\text{loop}}$	$\frac{-16\pi^2 C^{(1)}_{\alpha\beta\gamma}}{C^{(0)}_{\alpha\beta\gamma}}\|_{\text{mixing}}$	Sum
2A	5J	5J	6	2	2	$\frac{38}{5}$	$-\frac{3}{5}$	7
2B	5J	5J	0	2	2	$\frac{10}{7}$	$-\frac{15}{14}$	$\frac{5}{14}$
2B	5J	5K	0	2	0	2	$-\frac{5}{4}$	$\frac{3}{4}$

Table 5: The evaluated $\langle 2\|5\|5\rangle$ three-point correlators.

As reported in the introduction we make the general observation, that the radiative corrections to the three-point structure constants for a three-point function of two protected operators with one unprotected operator the structure constants follow the simple pattern:

$$\frac{C^{(1)}_{\alpha\beta\gamma}}{C^{(0)}_{\alpha\beta\gamma}}\Big|_{\text{loop}} = -\frac{1}{2}\,\gamma_\gamma \qquad \text{if } \gamma_\alpha = \gamma_\beta = 0\,. \tag{130}$$

This occurred in all applicable 17 cases we observed. Unfortunately this pattern does not survive once the mixing contributions are included.

6.5 Radiative contributions to $\langle\mathcal{K}|\mathcal{O}|\mathcal{O}\rangle$ correlators

In this subsection we derive a compact result for the radiative contributions to the three-point function of a Konishi operator with two arbitrary operators of same length from a diagonal basis. The three-point function then takes the general form

$$C^{(1)}_{\alpha\beta\mathcal{K}}\Big|_{\text{loop}} = -\left(\frac{\gamma_\alpha}{\Delta^{(0)}_\alpha} + \frac{\gamma_\beta}{\Delta^{(0)}_\beta} + \frac{\gamma_\mathcal{K}}{\Delta^{(0)}_\mathcal{K}}\right) C^{(0)}_{\alpha\beta\mathcal{K}} = -\frac{\delta_{\alpha\beta}}{4\pi^2\sqrt{3}}\left(2\gamma_\alpha + \frac{3}{8\pi^2}\Delta^{(0)}_\alpha\right)\,, \tag{131}$$

as already mentioned in the introduction.

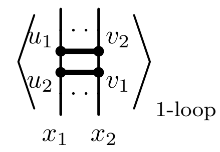
This may be shown as follows. Let $\mathcal{K}$ be the length two Konishi operator and the set $\{\mathcal{O}_\alpha\}$ an arbitrary non-diagonal basis for the operators of length $\Delta^{(0)}$ that can be written in terms of attached vectors, namely

$$\mathcal{K} = \frac{1}{\sqrt{12}}\sum_i \operatorname{Tr}\left(\phi^i\phi^i\right) \tag{132}$$

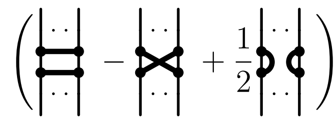
$$\mathcal{O}_\alpha = \operatorname{Tr}\left(u^\alpha_1\cdot\phi\cdots u^\alpha_{\Delta^{(0)}}\cdot\phi\right) \qquad\qquad (\Delta^{(0)}>2). \tag{133}$$

Let $Z_k \subset S_k$ denote the set of cyclic permutations of $(1,2,\ldots,k)$.

We choose the renormalization scheme $\varepsilon \to e\varepsilon$ in which only the 2-gons hold finite contributions



$$= I^2_{12}\,\frac{\lambda}{8\pi^2}\left(\ln\frac{\varepsilon^2}{x^2_{12}}+1\right) \tag{134}$$



while the 3-gons only contribute to the logarithmic terms. For the two-point functions we get

$$\langle\mathcal{O}_\alpha(x_1)\,\mathcal{O}_\beta(x_2)\rangle = I^{\Delta^{(0)}}_{12}\sum_{\sigma\in Z^{(0)}_\Delta}\left[\prod_{i=1}^{\Delta^{(0)}} u^\alpha_i\cdot u^\beta_{\sigma(i)} + \frac{\lambda}{8\pi^2}\left(\ln\frac{\varepsilon^2}{x^2_{12}}+1\right)\right.$$
$$\times\sum_{\tau\in Z^{(0)}_\Delta}\left(u^\alpha_{\tau(1)}\cdot u^\beta_{\tau\circ\sigma(1)}\,u^\alpha_{\tau(2)}\cdot u^\beta_{\tau\circ\sigma(2)} - u^\alpha_{\tau(1)}\cdot u^\beta_{\tau\circ\sigma(2)}\right.$$

$$
\times u^{\alpha}_{\tau(2)} \cdot u^{\beta}_{\tau\circ\sigma(1)} + \frac{1}{2}\, u^{\alpha}_{\tau(1)} \cdot u^{\alpha}_{\tau(2)}\, u^{\beta}_{\tau\circ\sigma(1)} \cdot u^{\beta}_{\tau\circ\sigma(2)}\Big)
\times \prod_{i=3}^{\Delta^{(0)}} u^{\alpha}_{\tau(i)} \cdot u^{\beta}_{\tau\circ\sigma(i)}\Bigg]. \tag{135}
$$

Now let $\mathcal{D}_\alpha = M_{\alpha\beta}\,\mathcal{O}_\beta$ denote a diagonal basis of the length $\Delta^{(0)}$ subspace. Then

$$
\langle \mathcal{D}_\alpha(x_1)\,\mathcal{D}_\beta(x_2)\rangle = \frac{1}{x_{12}^{2\Delta^{(0)}}}\left(\delta_{\alpha\beta} + \lambda g_{\alpha\beta} + \lambda\gamma_\alpha\delta_{\alpha\beta}\ln\frac{\varepsilon^2}{x_{12}^2}\right) = M_{\alpha\gamma}\,M_{\beta\delta}\,\langle \mathcal{O}_\gamma(x_1)\,\mathcal{O}_\delta(x_2)\rangle \tag{136}
$$

from which we immediately get the condition for tree-level diagonality

$$
\sum_{\sigma\in Z_\Delta^{(0)}} M_{\alpha\gamma}\,M_{\beta\delta}\prod_{i=1}^{\Delta^{(0)}} u_i^{\gamma}\cdot u^{\delta}_{\sigma(i)} = (2\pi)^{2\Delta^{(0)}}\,\delta_{\alpha\beta}. \tag{137}
$$

Using this result we obtain

$$
\begin{aligned}
\langle \mathcal{D}_\alpha(x_1)\,\mathcal{D}_\beta(x_2)\rangle = \frac{1}{x_{12}^{2\Delta^{(0)}}}\Bigg(\delta_{\alpha\beta} + \frac{\lambda}{8\pi^2}\left(\ln\frac{\varepsilon^2}{x_{12}^2}+1\right)\Bigg[\Delta^{(0)}\,\delta_{\alpha\beta} - \frac{1}{(2\pi)^{2\Delta^{(0)}}}\\
\times \sum_{\sigma\in Z_\Delta^{(0)}}\sum_{\tau\in Z_\Delta^{(0)}} M_{\alpha\gamma}\,M_{\beta\delta}\Big(u^{\gamma}_{\tau(1)}\cdot u^{\delta}_{\tau\circ\sigma(2)}\,u^{\gamma}_{\tau(2)}\cdot u^{\delta}_{\tau\circ\sigma(1)}\\
-\frac{1}{2}u^{\gamma}_{\tau(1)}\cdot u^{\gamma}_{\tau(2)}\,u^{\delta}_{\tau\circ\sigma(1)}\cdot u^{\delta}_{\tau\circ\sigma(2)}\Big)\times\prod_{i=3}^{\Delta^{(0)}} u^{\gamma}_{\tau(i)}\cdot u^{\delta}_{\tau\circ\sigma(i)}\Bigg]\Bigg)
\end{aligned} \tag{138}
$$

and thus the condition for one-loop diagonality

$$
\begin{aligned}
(2\pi)^{2\Delta^{(0)}}\,\delta_{\alpha\beta}\,\left(\Delta^{(0)} - 8\pi^2\,\gamma_\alpha\right) = \sum_{\sigma\in Z_\Delta^{(0)}}\sum_{\tau\in Z_\Delta^{(0)}} M_{\alpha\gamma}\,M_{\beta\delta}\Big(u^{\gamma}_{\tau(1)}\cdot u^{\delta}_{\tau\circ\sigma(2)}\,u^{\gamma}_{\tau(2)}\cdot u^{\delta}_{\tau\circ\sigma(1)}\\
-\frac{1}{2}u^{\gamma}_{\tau(1)}\cdot u^{\gamma}_{\tau(2)}\,u^{\delta}_{\tau\circ\sigma(1)}\cdot u^{\delta}_{\tau\circ\sigma(2)}\Big)\prod_{i=3}^{\Delta^{(0)}} u^{\gamma}_{\tau(i)}\cdot u^{\delta}_{\tau\circ\sigma(i)}
\end{aligned} \tag{139}
$$

and

$$
g_\alpha = \gamma_\alpha\,. \tag{140}
$$

The three-point functions are

$$
\begin{aligned}
&\langle \mathcal{D}_\alpha(x_1)\,\mathcal{D}_\beta(x_2)\mathcal{K}(x_3)\rangle = M_{\alpha\gamma}\,M_{\beta\delta}\,\langle \mathcal{O}_\alpha(x_1)\,\mathcal{O}_\beta(x_2)\mathcal{K}(x_3)\rangle \\
&\quad = \frac{1}{(2\pi)^{2\Delta^{(0)}+2}\sqrt{3}\,x_{12}^{2\Delta^{(0)}-2}\,x_{13}^2\,x_{23}^2} \sum_{\sigma\in Z_\Delta^{(0)}}\sum_{\tau\in Z_\Delta^{(0)}} M_{\alpha\gamma}\,M_{\beta\delta}\times\Bigg[\prod_{i=1}^{\Delta^{(0)}} u^\gamma_{\sigma(i)}\cdot u^\delta_{\tau(i)} \\
&\quad + \frac{\lambda}{8\pi^2}\sum_{\rho\in Z_{\Delta^{(0)}-2}}\Big(u^\gamma_{\sigma\circ\rho(1)}\cdot u^\delta_{\tau\circ\rho(1)}\,u^\gamma_{\sigma\circ\rho(2)}\cdot u^\delta_{\tau\circ\rho(2)} - u^\gamma_{\sigma\circ\rho(1)}\cdot u^\delta_{\tau\circ\rho(2)}\,u^\gamma_{\sigma\circ\rho(2)}\cdot u^\delta_{\tau\circ\rho(1)} \\
&\quad + \frac{1}{2}u^\gamma_{\sigma\circ\rho(1)}\cdot u^\gamma_{\sigma\circ\rho(2)}\,u^\delta_{\tau\circ\rho(1)}\cdot u^\delta_{\tau\circ\rho(2)}\Big)\times\prod_{i=3}^{\Delta^{(0)}-2}\Big(u^\gamma_{\sigma\circ\rho(i)}\cdot u^\delta_{\tau\circ\rho(i)}\Big) \\
&\quad \times u^\gamma_{\sigma(\Delta^{(0)}-1)}\cdot u^\delta_{\tau(\Delta^{(0)}-1)}\,u^\gamma_{\sigma(\Delta^{(0)})}\cdot u^\delta_{\tau(\Delta^{(0)})} + \lambda\times\text{logs}\Bigg] \\
&\quad \stackrel{!}{=} \frac{1}{x_{12}^{2\Delta^{(0)}-2}\,x_{13}^2\,x_{23}^2}\Big(C^{(0)}_{\alpha\beta\mathcal{K}} + \lambda\,\widetilde{C}^{(1)}_{\alpha\beta\mathcal{K}} + \lambda\times\text{logs}\Big)
\end{aligned}
\tag{141}
$$

and we obtain the tree-level structure constant

$$
\begin{aligned}
C^{(0)}_{\alpha\beta\mathcal{K}} &= \frac{1}{(2\pi)^{2\Delta^{(0)}+2}\sqrt{3}}\sum_{\sigma\in Z_\Delta^{(0)}}\sum_{\tau\in Z_\Delta^{(0)}} M_{\alpha\gamma}\,M_{\beta\delta}\prod_{i=1}^{\Delta^{(0)}} u^\gamma_{\sigma(i)}\cdot u^\delta_{\tau(i)} \\
&= \frac{\Delta^{(0)}}{(2\pi)^{2\Delta^{(0)}+2}\sqrt{3}}\sum_{\tau\in Z_\Delta^{(0)}} M_{\alpha\gamma}\,M_{\beta\delta}\prod_{i=1}^{\Delta^{(0)}} u^\gamma_{i}\cdot u^\delta_{\tau(i)},
\end{aligned}
\tag{142}
$$

where we omitted one sum over all permutations in the second line because the first sum already delivers all possible contractions.

Using equation (137) we get

$$
C^{(0)}_{\alpha\beta\mathcal{K}} = \frac{\Delta^{(0)}}{4\pi^2\sqrt{3}}\,\delta_{\alpha\beta}\,. \tag{143}
$$

The one-loop structure constant is

$$
\begin{aligned}
\widetilde{C}^{(1)}_{\alpha\beta\mathcal{K}} &= \frac{1}{(2\pi)^{2\Delta^{(0)}+4}\sqrt{12}}\sum_{\sigma\in Z_\Delta^{(0)}}\sum_{\tau\in Z_\Delta^{(0)}}\sum_{\rho\in Z_{\Delta^{(0)}-2}} M_{\alpha\gamma}\,M_{\beta\delta} \\
&\times\Bigg[\prod_{i=1}^{\Delta^{(0)}-2}\Big(u^\gamma_{\sigma\circ\rho(i)}\cdot u^\delta_{\tau\circ\rho(i)}\Big)\times u^\gamma_{\sigma(\Delta^{(0)}-1)}\cdot u^\delta_{\tau(\Delta^{(0)}-1)}\,u^\gamma_{\sigma(\Delta^{(0)})}\cdot u^\delta_{\tau(\Delta^{(0)})} \\
&-\Big(u^\gamma_{\sigma\circ\rho(1)}\cdot u^\delta_{\tau\circ\rho(2)}\,u^\gamma_{\sigma\circ\rho(2)}\cdot u^\delta_{\tau\circ\rho(1)} - \frac{1}{2}\,u^\gamma_{\sigma\circ\rho(1)}\cdot u^\gamma_{\sigma\circ\rho(2)}u^\delta_{\tau\circ\rho(1)}\cdot u^\delta_{\tau\circ\rho(2)}\Big) \\
&\times\prod_{i=3}^{\Delta^{(0)}-2}\Big(u^\gamma_{\sigma\circ\rho(i)}\cdot u^\delta_{\tau\circ\rho(i)}\Big)\times u^\gamma_{\sigma(\Delta^{(0)}-1)}\cdot u^\delta_{\tau(\Delta^{(0)}-1)}\,u^\gamma_{\sigma(\Delta^{(0)})}\cdot u^\delta_{\tau(\Delta^{(0)})}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\delta_{\alpha\beta}}{(2\pi)^4\sqrt{12}}\left[(\Delta^{(0)}-2)\,\Delta^{(0)} - (\Delta^{(0)}-2)\,(\Delta^{(0)}-8\pi^2\,\gamma_\alpha)\right] \\
&= \frac{(\Delta^{(0)}-2)\,\gamma_\alpha}{4\pi^2\sqrt{3}}\,\delta_{\alpha\beta},
\end{aligned} \tag{144}
$$

where the sum over the ρ-permutations gives only a factor of $(\Delta^{(0)}-2)$ and we made use of equations (137) and (139) in the second step.

The renormalization scheme independent structure constants

$$
C^{(1)}_{\alpha\beta\gamma} = \widetilde{C}^{(1)}_{\alpha\beta\gamma} - \frac{1}{2}\,C^{(0)}_{\alpha\beta\gamma}\,\left(g_\alpha + g_\beta + g_\gamma\right) \tag{145}
$$

may now be written down using (143), (144) and (140) to find

$$
C^{(1)}_{\alpha\beta\mathcal{K}} = \widetilde{C}^{(1)}_{\alpha\beta\mathcal{K}} - \frac{1}{2}\,C^{(0)}_{\alpha\beta\mathcal{K}}\,\left(\gamma_\alpha + \gamma_\beta + \frac{3}{4\pi^2}\right) = -\left(\frac{\gamma_\alpha}{\Delta^{(0)}_\alpha} + \frac{\gamma_\beta}{\Delta^{(0)}_\beta} + \frac{\gamma_\mathcal{K}}{\Delta^{(0)}_\mathcal{K}}\right)C^{(0)}_{\alpha\beta\mathcal{K}}\,. \tag{146}
$$

Acknowledgements

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A Conventions

In this Appendix we summarise our conventions. The Lagrangian and super-symmetry transformations of the four dimensional $\mathcal{N}=4$ SYM can be derived by dimensional reduction from the ten dimensional $\mathcal{N}=1$ SYM theory. We adopt the mostly-minus metric $(+,-,-,-)$ and the following conventions for the $SU(N)$ gauge group generators:

$$
\mathrm{Tr}\left(T^aT^b\right) = \frac{1}{2}\delta^{ab},\;\; [T^a,T^b] = i f^{ab}_c T^c,\;\; (T^a)^i_j (T^a)^k_l = \frac{1}{2}(\delta^i_l\delta^k_j - \frac{1}{N}\delta^i_j\delta^k_l) \tag{147}
$$

The Lagrangian reads

$$
\begin{aligned}
L = \mathrm{Tr}\Big[&-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + 2D_\mu\Phi_{AB}D^\mu\Phi^{AB} + 2i\psi^{\alpha A}\sigma^\mu_{\alpha\dot\alpha}(D_\mu\bar\psi_A)^{\dot\alpha} + \\
2\,g^2_{YM}\,[\Phi^{AB},\Phi^{CD}][\Phi_{AB},\Phi_{CD}] &- 2\sqrt{2}\,g_{YM}([\psi^{\alpha A},\Phi_{AB}]\psi^B_\alpha - [\bar\psi_{\dot\alpha A},\Phi^{AB}]\bar\psi^{\dot\alpha}_B)\Big],
\end{aligned} \tag{148}
$$

where Φ_{AB} denote the six complex scalar fields of $\mathcal{N}=4$ SYM which satisfy $\Phi^{AB} = \frac{1}{2}\epsilon_{ABCD}\,\Phi^{CD} = \bar\Phi_{AB}$. Sometimes it is more convenient to work with three complex scalar fields Z_1, Z_2, Z_3 and their complex conjugates defined as follows

$$
\begin{aligned}
Z_1 = 2\,\Phi_{14}\,, &\quad \bar Z_1 = 2\,\Phi_{23} = 2\,\Phi^{14} \\
Z_2 = 2\,\Phi_{24}\,, &\quad \bar Z_2 = 2\,\Phi_{31} = 2\,\Phi^{24} \\
Z_3 = 2\,\Phi_{34}\,, &\quad \bar Z_3 = 2\,\Phi_{12} = 2\,\Phi^{34}\,.
\end{aligned} \tag{149}
$$

with $Z_1 = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, $Z_2 = \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4)$, $Z = Z_3 = \frac{1}{\sqrt{2}}(\phi_5 + i\phi_6)$.

For the propagators we note

$$\begin{aligned} \langle Z_i(x)^a{}_b\, \bar{Z}_j(y)^c{}_d \rangle &= \frac{1}{2}\, \delta_{ij}\, \delta^c_b \delta^a_d\, \varDelta_{xy} \qquad a,b,c,d = 1,\ldots,N, \quad i,j = 1,2,3\,, \\ \langle \psi^A_\alpha(x)^a{}_b\, \bar{\psi}_{\dot\alpha\, B}(y)^c{}_d \rangle &= \frac{i}{2}\, \delta^A_B\, \sigma^\mu_{\alpha\dot\alpha}\, \partial^x_\mu \varDelta_{xy}\,, \end{aligned} \tag{150}$$

with $\varDelta_{xy} = -\frac{1}{4\pi^2\,(x-y)^2}$. From these one deduces

$$\begin{aligned} \langle \partial_\mu Z_i(x)^a{}_b\, \bar{Z}_j(y)^c{}_d \rangle &= \frac{1}{2}\, \delta_{ij}\, \delta^c_b \delta^a_d\, \partial^x_\mu \varDelta_{xy}\,, \\ \partial^1_\mu \varDelta_{12}\, \partial^{1\,\mu} \varDelta_{13} &= -8\pi^2\, (\varDelta_{12}\varDelta_{13}^2 + \varDelta_{12}^2\varDelta_{13} - \varDelta_{12}^2\, \varDelta_{13}^2\, \varDelta_{23}^{-1})\,. \end{aligned} \tag{151}$$

Moreover, one may derive an effective spinor index free contraction of the gluinos

$$\langle \psi^A(x)^a_b\, \bar{\psi}_B(y)^c_d \rangle_{\text{effective}} = i\,\sqrt{2}\, 2\pi\, \varDelta^{3/2}_{xy}\, \delta^A_B\, \delta^a_d\, \delta^c_b\,, \tag{152}$$

which appears in correlators involving only two bi-fermion insertions of the form

$$\langle \psi^{A_1\,\alpha}(x)^{a_1}_{b_1}\, \psi^{A_2}_\alpha(x)^{a_2}_{b_2}\, \bar{\psi}_{B_1\,\dot\alpha}(y)^{c_1}_{d_1}\, \bar{\psi}^{\dot\alpha}_{B_2}(y)^{c_2}_{d_2} \rangle\,, \tag{153}$$

which are spinor-index singlets and are of relevance in the computations at hand.

We report here the form of currents associated to the superconformal transformations of $\mathcal{N} = 4$ SYM (see Appendix A of [29]):

$$\begin{aligned} \bar{S}^{\mu\dot\alpha}_{\;\;A} = 2x_\tau (\bar\sigma^\tau)^{\dot\alpha\alpha} \mathrm{Tr}\Big(&(\sigma^{\rho\nu})^\beta_\alpha F_{\rho\nu} \sigma^\mu_{\beta\dot\beta} \bar\psi^{\dot\beta}_A + 2\sqrt{2} D_\rho \varPhi_{AB} \sigma^\rho_{\alpha\dot\alpha} \bar\sigma^{\mu\,\dot\alpha\beta} \psi^B_\beta + \\ &-4\mathrm{i}g[\varPhi_{AC}, \varPhi^{CB}] \sigma^\mu_{\alpha\dot\beta} \bar\psi^{\dot\beta}_B \Big) + 8\sqrt{2}\mathrm{Tr}\left(\phi_{AB} (\bar\sigma^\mu)^{\dot\alpha\alpha} \psi^B_\alpha\right), \end{aligned} \tag{154a}$$

$$\begin{aligned} S^{\mu A}_{\;\;\alpha} = 2x_\tau \sigma^\tau_{\alpha\dot\alpha} \mathrm{Tr}\Big(&(\bar\sigma^{\rho\nu})^{\dot\alpha}_{\dot\beta} F_{\rho\nu} \bar\sigma^{\mu\,\dot\beta\beta} \psi^A_\beta - 2\sqrt{2} D_\rho \varPhi^{AB} \bar\sigma^{\rho\,\dot\alpha\alpha} \sigma^\mu_{\alpha\dot\beta} \bar\psi^{\dot\beta}_B + \\ &-4\mathrm{i}g[\varPhi^{AC}, \varPhi_{CB}] \bar\sigma^{\mu\dot\alpha\alpha} \psi^B_\alpha \Big) - 8\sqrt{2}\mathrm{Tr}\left(\phi^{AB} \sigma^\mu_{\alpha\dot\alpha} \bar\psi^{\dot\alpha}_B\right), \end{aligned} \tag{154b}$$

from which one can derive the tree-level and order-g_{YM} superconformal variation of the fields. In particular, we will use:

- the tree-level superconformal variation of a single fermion [29]:

$$\bar{S}^{\dot\alpha}_A \bar\psi_{B\dot\beta} = 4\sqrt{2} i \varPhi_{AB} \delta^{\dot\alpha}_{\dot\beta}; \tag{155}$$

- the order-g_{YM} superconformal variation of a pair of scalar fields [29]:

$$\bar{S}^{\dot\alpha}_A \varPhi_{BC} \varPhi_{DE}(0) = -\mathrm{i}\, \frac{gN}{32\pi^2} \left(\epsilon_{ABC[D} \bar\psi_{E]}(0) - \epsilon_{ADE[B} \bar\psi^{\dot\alpha}_{C]}(0) \right)\,, \tag{156}$$

  where $\epsilon_{ABC[D}\bar\psi_{E]} = \frac{1}{2}(\epsilon_{ABCD}\bar\psi_E - \epsilon_{ABCE}\bar\psi_D)$.

B Normalization of the states

In this section we compute explicitly the order-g_{YM}^2 contribution to the normalization of the non-BPS operators coming from the mixing. We require that the two-point functions are canonically normalized:

$$\langle \bar{\hat{\mathcal{O}}}(x)\hat{\mathcal{O}}(y)\rangle = \frac{(-1)^L}{[(x-y)^2]^L}, \tag{157}$$

where L is the operator length.

Let us start from the Highest Weight State (44), focusing first on the treelevel contribution coming from the leading term. Using:

$$\sum_{p=0}^{J} \cos\frac{\pi n(2p+3)}{J+3} = -2\cos\frac{\pi n}{J+3} \qquad \text{and} \qquad \sum_{p=0}^{J} \cos^2\frac{\pi n(2p+3)}{J+3} = \frac{J+3}{2} - 2\cos^2\frac{\pi n}{J+3}, \tag{158}$$

it is straightforward to show that

$$\sum_{p,q=0}^{J} \cos\frac{\pi n(2p+3)}{J+3}\cos\frac{\pi n(2q+3)}{J+3}\times$$
$$\langle \mathrm{Tr}\left(\Phi_{AB}\bar{Z}^p\Phi^{AB}\bar{Z}^{J-p}\right)\mathrm{Tr}\left(\Phi_{AB}Z^q\Phi^{AB}Z^{J-q}\right)\rangle \;=\; (J+3)\Big(\frac{N}{8\pi^2}\Big)^{J+2}\frac{(-1)^{J+2}}{[(x-y)^2]^{J+2}}, \tag{159}$$

from which we get the leading term in (45).

The fermionic terms contribute to the normalization at order g^2 via tree-level contractions. So we shall compute

$$\sum_{p,q=0}^{J-1} \sin\frac{\pi n(2p+4)}{J+3}\sin\frac{\pi n(2q+4)}{J+3}\times$$
$$\Big[\langle \mathrm{Tr}\left(\bar\psi_{1\dot\alpha}\bar{Z}^p\bar\psi_2^{\dot\alpha}\bar{Z}^{J-p-1}\right)(x)\mathrm{Tr}\left(\psi^{1\alpha}Z^q\psi^2_\alpha Z^{J-q-1}\right)(y)\rangle +$$
$$\langle \mathrm{Tr}\left(\psi^{3\alpha}Z^p\psi^4_\alpha Z^{J-p-1}\right)(x)\mathrm{Tr}\left(\bar\psi_{3\dot\alpha}\bar{Z}^q\bar\psi_4^{\dot\alpha}\bar{Z}^{J-q-1}\right)(y)\rangle\Big] \tag{160}$$

The first term within squared brackets yields:

$$\langle \mathrm{Tr}\left(\bar\psi_{1\dot\alpha}\bar{Z}^p\bar\psi_2^{\dot\alpha}\bar{Z}^{J-p-1}\right)(x)\mathrm{Tr}\left(\psi^{1\alpha}Z^q\psi^2_\alpha Z^{J-q-1}\right)(y)\rangle = \left(\frac{N}{2}\right)^{J+1} 32\pi^2\Delta_{xy}^{J+2}\delta_{q,J-p-1} \tag{161}$$

while the second term just doubles this result.

So, if we includes the coefficients in front to the fermionic term in (44), we get:

$$-\mathcal{N}^2\frac{N^2}{(8\sqrt{2}\pi^2)^2}\sin^2\frac{\pi n}{J+3}\sum_{p=0}^{J-1}\sin^2\frac{\pi n(2p+4)}{J+3}\left(\frac{N}{2}\right)^{J+1}64\pi^2\Delta_{xy}^{J+2}\delta_{q,J-p-1} \tag{162}$$

The sum yelds:

$$\sum_{p=0}^{J-1} \sin^2 \frac{\pi n(2p+4)}{J+3} = \frac{J-1}{2} + 2\cos^2 \frac{2\pi n}{J+3} \tag{163}$$

Putting everything together, the finite part of the two point function up to order g^2 is:

$$\langle \bar{\hat{\mathcal{O}}}{}_n^J(x) \hat{\mathcal{O}}_n^J(y) \rangle = \\ \mathcal{N}^2 (J+3) \Big(\frac{N}{8\pi^2}\Big)^{J+2} \Big[1 - \frac{g^2 N}{\pi^2 (J+3)} \sin^2 \frac{\pi n}{J+3} \left(\frac{J-1}{2} + 2\cos^2 \frac{2\pi n}{J+3} \right) \Big] \frac{(-1)^{J+2}}{[(x-y)^2]^{J+2}}. \tag{164}$$

Requiring this being canonically normalized:

$$\langle \bar{\hat{\mathcal{O}}}{}_n^J(x) \hat{\mathcal{O}}_n^J(y) \rangle = \frac{(-1)^{J+2}}{[(x-y)^2]^{J+2}}, \tag{165}$$

we get ($N_0 = \frac{N}{8\pi^2}$)

$$\mathcal{N} = \sqrt{\frac{N_0^{-J-2}}{J+3}} \left[1 - \frac{g^2 N}{\pi^2 (J+3)} \sin^2 \frac{\pi n}{J+3} \left(\frac{J-1}{2} + 2\cos^2 \frac{2\pi n}{J+3} \right) \right]^{-\frac{1}{2}} \tag{166}$$

which, expanded for small g, gives the result in (45).

Now let us focus on the operators $\hat{\mathcal{O}}_{4A}$ and $\hat{\mathcal{O}}_{4E}$. The tree level contribution can be rewritten as:

$$\left(\frac{8\pi^2}{N}\right)^4 \mathcal{N}_{A/E}^2 \Big[\langle \mathrm{Tr}\left(\Phi_{AB}\Phi^{AB}\Phi_{CD}\Phi^{CD} \right)(x) \mathrm{Tr}\left(\Phi_{A'B'}\Phi^{A'B'}\Phi_{C'D'}\Phi^{C'D'} \right)(y) \rangle \\ + 2\alpha_{A/E} \langle \mathrm{Tr}\left(\Phi_{AB}\Phi^{AB}\Phi_{CD}\Phi^{CD} \right)(x) \mathrm{Tr}\left(\Phi_{A'B'}\Phi_{C'D'}\Phi^{A'B'}\Phi^{C'D'} \right)(y) \rangle \\ + \alpha_{A/E}^2 \langle \mathrm{Tr}\left(\Phi_{AB}\Phi_{CD}\Phi^{AB}\Phi^{CD} \right)(x) \mathrm{Tr}\left(\Phi_{A'B'}\Phi_{C'D'}\Phi^{A'B'}\Phi^{C'D'} \right)(y) \rangle \Big] \tag{167}$$

We recall that in the $SU(4)$ notation the correlator between the scalar fields reads:

$$\langle \Phi_{AB}(x)^a \Phi_{CD}(y)^b \rangle = \delta^{ab} \epsilon_{ABCD} \Delta_{xy} \qquad \text{and} \qquad \Phi^{AB} = \frac{1}{2} \epsilon^{ABCD} \Phi_{CD} \tag{168}$$

The computation of the different terms in (167) is then straigthforward and it yields:

$$\mathcal{N}_{A/E}^2 \left[\frac{21}{4} + 3\,\alpha_{A/E} + 9\,\alpha_{A/E}^2 \right] \frac{(-1)^4}{[(x-y)^2]^4} = \mathcal{N}_{A/E}^2 \frac{738 \mp 102\sqrt{41}}{16} \frac{(-1)^4}{[(x-y)^2]^4} \tag{169}$$

where we have replaced $\alpha_{A/E} = \frac{5 \mp \sqrt{41}}{4}$.

The contribution of the fermionic subleading terms is:

$$\left(\frac{8\pi^2}{N}\right)^4 \mathcal{N}_{A/E}^2 \frac{g^2 N^2}{(16\sqrt{2}\pi^2)^2} \frac{(3 \mp \sqrt{41})^2}{16} \left[\langle \mathrm{Tr}\left(\Phi^{AB} \bar{\psi}_{A\dot{\alpha}} \bar{\psi}_B^{\dot{\alpha}} \right)(x) \mathrm{Tr}\left(\Phi_{CD} \psi^{C\alpha} \psi_\alpha^D \right)(y) \rangle + \right. \\ \left. \langle \mathrm{Tr}\left(\Phi_{AB} \psi^{A\alpha} \psi_\alpha^B \right)(x) \mathrm{Tr}\left(\Phi^{CD} \bar{\psi}_{C\dot{\alpha}} \bar{\psi}_D^{\dot{\alpha}} \right)(y) \rangle \right] \tag{170}$$

Each term in the last formula gives the same result, namely:

$$\langle \mathrm{Tr}\left(\Phi^{AB}\bar{\psi}_{A\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}_{B}\right)(x)\mathrm{Tr}\left(\Phi_{CD}\psi^{C\alpha}\psi^{D}_{\alpha}\right)(y)\rangle = 3\Big(\frac{N}{2}\Big)^3 \Delta_{xy}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^{\nu}_{\alpha\dot{\alpha}}\partial^{(x)}_{\mu}\Delta_{xy}\partial^{(x)}_{\nu}\Delta_{xy} \tag{171}$$

But $\bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^{\nu}_{\alpha\dot{\alpha}} = 2\eta^{\mu\nu} + [\mu,\nu]$, where $[\mu,\nu]$ denotes a term antisymmetric in μ and ν which does not contribute to the final result, and $\partial^{(x)}_{\mu}\Delta_{xy}\partial^{\mu\,(x)}\Delta_{xy} = -16\pi^2\Delta^3_{xy}$, so all togheter the contribution of the fermionic subleading mixing term is:

$$-\mathcal{N}^2_{A/E}\frac{g^2_{YM}N}{(8\pi)^2}3(3\mp\sqrt{41})^2\frac{(-1)^4}{[(x-y)^2]^4} \tag{172}$$

and then the finite part of the two point function up to order g^2_{YM} is:

$$\langle\bar{\hat{\mathcal{O}}}_{4A/E}(x)\hat{\mathcal{O}}_{4A/E}\rangle = \mathcal{N}^2_{A/E}\Big[\frac{738-102\sqrt{41}}{16} - \frac{g^2_{YM}N}{(8\pi)^2}3(50\mp 6\sqrt{41})\Big]\frac{(-1)^4}{[(x-y)^2]^4} \tag{173}$$

Requiring that the above correlator is canonically normalized we get:

$$\mathcal{N}_{A/E} = \frac{4}{\sqrt{738-102\sqrt{41}}}\Big[1+\frac{g^2_{YM}N}{4\pi^2}\frac{25\mp 3\sqrt{41}}{246\mp 34\sqrt{41}}\Big] \tag{174}$$

C Cancellation of the terms violating conformal invariance

In the correlators we computed in section 6, the terms coming from a Yukawa insertion which do not respect conformal invariance cancel out agagainst similar contributions from the bi-derivative mixing terms. This behavour has already been shown in [26], where the contribution to the one loop structure constant has been computed for a class of correlators involving the highest weight state in (44).

In this section, we are going to show explicitly the same cancellation occurring in the classes of correlators $\langle 3B|3B|4A\rangle$ and $\langle 3B|3B|4E\rangle$.

Thus let us consider

$$\langle\hat{\mathcal{O}}^1_1(x_1)\bar{\hat{\mathcal{O}}}^1_1(x_2)\hat{\mathcal{O}}_{4A/E}(x_3)\rangle, \tag{175}$$

and drop the overall normalization, which does not play any role in this computation. So we are going to take:

$$\hat{\mathcal{O}}^1_1 = \mathrm{Tr}\left(\Phi_{AB}\Phi^{AB}Z\right) \tag{176}$$

and write

$$\hat{\mathcal{O}}_{4A/E} = \mathcal{O}_{scal} + \mathcal{O}_{\psi\psi} + \mathcal{O}_{DD},$$

where

$$\mathcal{O}_{\psi\psi} = -g\mathcal{N}\frac{3\mp\sqrt{41}}{4}\Big[\mathrm{Tr}\left(\Phi_{AB}\psi^{A\alpha}\psi^{B}_{\alpha}\right) - \mathrm{Tr}\left(\Phi^{AB}\bar{\psi}_{A\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}_{B}\right)\Big] \tag{177}$$

$$\mathcal{O}_{DD} = g^2\mathcal{N}^2(3\mp\sqrt{41})\mathrm{Tr}\left(D_\mu\Phi_{AB}D^\mu\Phi^{AB}\right) \tag{178}$$

with $\mathcal{N} = \frac{N}{16\sqrt{2}\pi^2}$. We want to show that in the sum $\langle\hat{\mathcal{O}}^1_1\bar{\hat{\mathcal{O}}}^1_1\mathcal{O}_{\psi\psi}\rangle + \langle\hat{\mathcal{O}}^1_1\bar{\hat{\mathcal{O}}}^1_1\mathcal{O}_{DD}\rangle$ all the terms which do not respect conformal invariance cancel out.

Let us start from $\langle\hat{\mathcal{O}}^1_1\bar{\hat{\mathcal{O}}}^1_1\mathcal{O}_{\psi\psi}\rangle$. It is sufficient to focus on the term with unbarred fermions, because the other term will just double this result. The Yukawa coupling relevant for this computation is:

$$\mathrm{i}4\sqrt{2}g\int d^4w\mathrm{Tr}\left(\Phi^{XY}\bar{\psi}^{\dot{\alpha}}_X\bar{\psi}^{\dot{\beta}}_Y\right)\epsilon_{\dot{\alpha}\dot{\beta}} \tag{179}$$

We can contract the scalar in the Yukawa with any scalar field in the operator in x_1, and this forces the contraction of the remaining scalar in longer operator with a scalar in x_2. Obviously, one must also consider the opposite situation, where the scalar of the Yukawa is contracted with a scalar in x_2, and hence the remaining scalar in $\mathcal{O}_{DD}$ with one in x_1. However, this just exchange the role of x_1 and x_2 in the result, so, by now, it is sufficient to focus on the first case, and then add the same result with x_1 and x_2 swapped.

For each of the three diagrams we get contracting the Yukawa scalar with a specific scalar in x_1, we can further contract the remaining six scalar in three possible ways. Taking into account all the diagrams, and then summing the result with $x_1 \leftrightarrow x_2$ one gets:

$$\langle\hat{\mathcal{O}}^1_1\bar{\hat{\mathcal{O}}}^1_1\mathcal{O}_{\psi\psi}\rangle = -\mathrm{i}g^2\frac{N^4}{8\sqrt{2}}\mathcal{N}3(3\mp\sqrt{41})\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta}\bar{\sigma}^{\mu\dot{\beta}\alpha}\bar{\sigma}^{\nu\dot{\alpha}\beta}$$
$$\Delta^2_{12}\int d^4w\partial^{(w)}_\mu\Delta_{w3}\partial^{(w)}_\nu\Delta_{w3}\big[\Delta_{2w}\Delta_{13} + \Delta_{1w}\Delta_{23}\big]. \tag{180}$$

However $\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta}\bar{\sigma}^{\mu\dot{\beta}\alpha}\bar{\sigma}^{\nu\dot{\alpha}\beta} = -2\eta^{\mu\nu}$ plus terms antisymmetric in μ and ν, and then the integrals become:

$$\int d^4w\Delta_{2w}\partial^{(w)}_\mu\Delta_{w3}\partial^{(w)}_\nu\Delta_{w3} = -\frac{\mathrm{i}}{2}\Delta^2_{23}, \tag{181}$$

$$\int d^4w\Delta_{1w}\partial^{(w)}_\mu\Delta_{w3}\partial^{(w)}_\nu\Delta_{w3} = -\frac{\mathrm{i}}{2}\Delta^2_{13}. \tag{182}$$

Substituting $\mathcal{N} = \frac{N}{16\sqrt{2}\pi^2}$ we get:

$$\langle\hat{\mathcal{O}}^1_1\bar{\hat{\mathcal{O}}}^1_1\mathcal{O}_{\psi\psi}\rangle = g^2_{YM}\frac{N^5}{(16\sqrt{2})^2\pi^2}3(3\mp\sqrt{41})\Delta^2_{12}\big[\Delta_{13}\Delta^2_{23} + \Delta^2_{13}\Delta_{23}\big]. \tag{183}$$

Moving to the computation of the derivative terms, one must contract one $D_\mu\Phi_{AB}$ with any of the scalars in x_1 and the other one with any of the scalar in x_2. This can be done in nine

independent ways. Then one must exchange the role of x_1 and x_2. However this would just double the result of the former case. Thus, summing all the diagrams one gets:

$$\langle \hat{\mathcal{O}}_1^1 \bar{\hat{\mathcal{O}}}_1^1 \mathcal{O}_{DD} \rangle = g^2 \frac{N^3}{8} \mathcal{N}^2 3(3 \mp \sqrt{41}) \Delta_{12}^2 \partial_\mu^{(x_3)} \Delta_{23} \partial^{\mu\,(x_3)} \Delta_{13} \tag{184}$$

Since $\partial_\mu^{(x_3)} \Delta_{23} \partial^{\mu\,(x_3)} \Delta_{13} = -8\pi^2 (\Delta_{13}\Delta_{23}^2 + \Delta_{13}^2 \Delta_{23} - \Delta_{13}^2 \Delta_{23}^2 \Delta_{12}^{-1})$, and replacing $\mathcal{N} = \frac{N}{16\sqrt{2}\pi^2}$, one finally gets:

$$\langle \hat{\mathcal{O}}_1^1 \bar{\hat{\mathcal{O}}}_1^1 \mathcal{O}_{DD} \rangle = -g^2 \frac{N^5}{(16\sqrt{2})^2 \pi^2} 3(3 \mp \sqrt{41}) \Delta_{12}^2 \left[\Delta_{13}\Delta_{23}^2 + \Delta_{13}^2 \Delta_{23} - \Delta_{13}^2 \Delta_{23}^2 \Delta_{12}^{-1} \right]. \tag{185}$$

Comparing (185) with (183), one notice that the first and the second term in (185) cancel out against (183), and that, up to the overall normalization, the contribution of the bi-fermion and bi-derivative mixing terms to the correlator reduces to:

$$g^2 \frac{N^5}{(16\sqrt{2})^2 \pi^2} 3(3 \mp \sqrt{41}) \Delta_{12} \Delta_{13}^2 \Delta_{23}^2, \tag{186}$$

in agreement with conformal invariance prescriptions.

References

[1] J. A. Minahan and K. Zarembo, *"The Bethe-ansatz for $\mathcal{N}=4$ super Yang-Mills"*, JHEP 0303, 013 (2003), hep-th/0212208.

[2] N. Beisert, C. Kristjansen and M. Staudacher, *"The dilatation operator of $N=4$ super Yang-Mills theory"*, Nucl. Phys. B664, 131 (2003), hep-th/0303060. • I. Bena, J. Polchinski and R. Roiban, *"Hidden symmetries of the $AdS_5 \times S^5$ superstring"*, Phys. Rev. D69, 046002 (2004), hep-th/0305116.

[3] N. Beisert and M. Staudacher, *"The $\mathcal{N}=4$ SYM Integrable Super Spin Chain"*, Nucl. Phys. B670, 439 (2003), hep-th/0307042.

[4] J. M. Maldacena, *"The large N limit of superconformal field theories and supergravity"*, Adv. Theor. Math. Phys. 2, 231 (1998), hep-th/9711200. • E. Witten, *"Anti-de Sitter space and holography"*, Adv. Theor. Math. Phys. 2, 253 (1998), hep-th/9802150. • S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *"Gauge theory correlators from non-critical string theory"*, Phys. Lett. B428, 105 (1998), hep-th/9802109.

[5] L. Brink, J. H. Schwarz and J. Scherk, *"Supersymmetric Yang-Mills Theories"*, Nucl. Phys. B121, 77 (1977). • F. Gliozzi, J. Scherk and D. I. Olive, *"Supersymmetry, Supergravity Theories and the Dual Spinor Model"*, Nucl. Phys. B122, 253 (1977).

[6] N. Beisert, *"The SU(2|3) Dynamic Spin Chain"*, Nucl. Phys. B682, 487 (2004), hep-th/0310252.

[7] M. Staudacher, *"The factorized S-matrix of CFT/AdS"*, JHEP 0505, 054 (2005), hep-th/0412188. • N. Beisert, *"The su(2|2) dynamic S-matrix"*, Adv. Theor. Math. Phys. 12, 945 (2008), hep-th/0511082.

[8] A. A. Tseytlin, *"Semiclassical strings in $AdS_5 \times S^5$ and scalar operators in $\mathcal{N} = 4$ SYM theory"*, Comptes Rendus Physique 5, 1049 (2004), hep-th/0407218. • A. V. Belitsky, V. M. Braun, A. S. Gorsky and G. P. Korchemsky, *"Integrability in QCD and beyond"*, Int. J. Mod. Phys. A19, 4715 (2004), hep-th/0407232. • K. Zarembo, *"Semiclassical Bethe ansatz and AdS/CFT"*, Comptes Rendus Physique 5, 1081 (2004), hep-th/0411191. • J. Plefka, *"Spinning strings and integrable spin chains in the AdS/CFT correspondence"*, Living. Rev. Relativity 8, 9 (2005), hep-th/0507136. • J. A. Minahan, *"A brief introduction to the Bethe ansatz in $\mathcal{N} = 4$ super-Yang-Mills"*, J. Phys. A39, 12657 (2006). • G. Arutyunov and S. Frolov, *"Foundations of the $AdS_5 \times S^5$ Superstring. Part I"*, J. Phys. A42, 254003 (2009), arxiv:0901.4937. • N. Beisert, *"Higher-Loop Integrability in $\mathcal{N} = 4$ Gauge Theory"*, Comptes Rendus Physique 5, 1039 (2004), hep-th/0409147.

[9] N. Beisert, *"The Dilatation Operator of $\mathcal{N} = 4$ Super Yang-Mills Theory and Integrability"*, Phys. Rept. 405, 1 (2004), hep-th/0407277.

[10] N. Beisert et al., *"Review of AdS/CFT Integrability: An Overview"*, arxiv:1012.3982.

[11] J. Ambjorn, R. A. Janik and C. Kristjansen, *"Wrapping interactions and a new source of corrections to the spin-chain / string duality"*, Nucl. Phys. B736, 288 (2006), hep-th/0510171.

[12] Z. Bajnok, A. Hegedus, R. A. Janik and T. Lukowski, *"Five loop Konishi from AdS/CFT"*, Nucl. Phys. B827, 426 (2010), arxiv:0906.4062. • F. Fiamberti, A. Santambrogio and C. Sieg, *"Five-loop anomalous dimension at critical wrapping order in N=4 SYM"*, JHEP 1003, 103 (2010), arxiv:0908.0234.

[13] N. Gromov, V. Kazakov, A. Kozak and P. Vieira, *"Exact Spectrum of Anomalous Dimensions of Planar $N = 4$ Supersymmetric Yang-Mills Theory: TBA and excited states"*, Lett. Math. Phys. 91, 265 (2010), arxiv:0902.4458. • D. Bombardelli, D. Fioravanti and R. Tateo, *"Thermodynamic Bethe Ansatz for planar AdS/CFT: a proposal"*, J. Phys. A42, 375401 (2009), arxiv:0902.3930. • G. Arutyunov and S. Frolov, *"Thermodynamic Bethe Ansatz for the $AdS_5 \times S^5$ Mirror Model"*, JHEP 0905, 068 (2009), arxiv:0903.0141.

[14] N. Gromov, V. Kazakov and P. Vieira, *"Exact Spectrum of Planar $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory: Konishi Dimension at Any Coupling"*, Phys. Rev. Lett. 104, 211601 (2010), arxiv:0906.4240. • S. Frolov, *"Konishi operator at intermediate coupling"*, arxiv:1006.5032.

[15] J. K. Erickson, G. W. Semenoff and K. Zarembo, *"Wilson loops in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory"*, Nucl. Phys. B582, 155 (2000), hep-th/0003055. • N. Drukker and D. J. Gross, *"An exact prediction of $\mathcal{N} = 4$ SUSYM theory for string theory"*, J. Math. Phys. 42, 2896 (2001), hep-th/0010274.

[16] Z. Bern, L. J. Dixon and V. A. Smirnov, *"Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond"*, Phys. Rev. D72, 085001 (2005), hep-th/0505205. • J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, *"Dual superconformal symmetry of scattering amplitudes in $\mathcal{N} = 4$ super-Yang–Mills theory"*, Nucl. Phys. B828, 317 (2010), arxiv:0807.1095.

[17] L. F. Alday and J. M. Maldacena, *"Gluon scattering amplitudes at strong coupling"*, JHEP 0706, 064 (2007), arxiv:0705.0303.

[18] L. F. Alday and R. Roiban, *"Scattering Amplitudes, Wilson Loops and the String/Gauge Theory Correspondence"*, Phys. Rept. 468, 153 (2008), arxiv:0807.1889. • J. M. Henn, *"Duality between Wilson loops and gluon amplitudes"*, Fortschr. Phys. 57, 729 (2009), arxiv:0903.0522.

[19] B. Eden, P. S. Howe and P. C. West, *"Nilpotent invariants in $N = 4$ SYM"*, Phys. Lett. B463, 19 (1999), hep-th/9905085. • G. Arutyunov, B. Eden and E. Sokatchev, *"On non-renormalization and OPE in superconformal field theories"*, Nucl. Phys. B619, 359 (2001), hep-th/0105254. • P. J. Heslop and P. S. Howe, *"OPEs and 3-point correlators of protected operators in $N = 4$ SYM"*, Nucl. Phys. B626, 265 (2002), hep-th/0107212. • S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, *"Three-point functions of chiral operators in $D = 4$, $N = 4$ SYM at large N"*, Adv. Theor. Math. Phys. 2, 697 (1998), hep-th/9806074. • A. Basu, M. B. Green and S. Sethi, *"Some systematics of the coupling constant dependence of $N = 4$ Yang-Mills"*, JHEP 0409, 045 (2004), hep-th/0406231.

[20] M. Bianchi, S. Kovacs, G. Rossi and Y. S. Stanev, *"Properties of the Konishi multiplet in $N = 4$ SYM theory"*, JHEP 0105, 042 (2001), hep-th/0104016.

[21] N. Beisert, C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, *"BMN Correlators and Operator Mixing in $\mathcal{N} = 4$ Super Yang-Mills Theory"*, Nucl. Phys. B650, 125 (2003), hep-th/0208178.

[22] R. Roiban and A. Volovich, *"Yang-Mills correlation functions from integrable spin chains"*, JHEP 0409, 032 (2004), hep-th/0407140.

[23] K. Okuyama and L.-S. Tseng, *"Three-point functions in $N = 4$ SYM theory at one-loop"*, JHEP 0408, 055 (2004), hep-th/0404190.

[24] L. F. Alday, J. R. David, E. Gava and K. S. Narain, *"Structure constants of planar $N = 4$ Yang Mills at one loop"*, JHEP 0509, 070 (2005), hep-th/0502186.

[25] L. F. Alday, J. R. David, E. Gava and K. S. Narain, *"Towards a string bit formulation of $N = 4$ super Yang- Mills"*, JHEP 0604, 014 (2006), hep-th/0510264.

[26] G. Georgiou, V. L. Gili and R. Russo, *"Operator mixing and three-point functions in N=4 SYM"*, JHEP 0910, 009 (2009), arxiv:0907.1567.

[27] G. Arutyunov, S. Frolov and A. Petkou, *"Perturbative and instanton corrections to the OPE of CPOs in $N = 4$ SYM(4)"*, Nucl. Phys. B602, 238 (2001), hep-th/0010137.

[28] C.-S. Chu, V. V. Khoze and G. Travaglini, *"Three-point functions in $N = 4$ Yang-Mills theory and pp- waves"*, JHEP 0206, 011 (2002), hep-th/0206005.

[29] G. Georgiou, V. L. Gili and R. Russo, *"Operator Mixing and the AdS/CFT correspondence"*, JHEP 0901, 082 (2009), arxiv:0810.0499.

[30] P. Y. Casteill, R. A. Janik, A. Jarosz and C. Kristjansen, *"Quasilocality of joining/splitting strings from coherent states"*, JHEP 0712, 069 (2007), arxiv:0710.4166.

[31] T. Yoneya, *"Holography in the large J limit of AdS/CFT correspondence and its applications"*, Prog. Theor. Phys. Suppl. 164, 82 (2007), hep-th/0607046. • S. Dobashi and T. Yoneya, *"Resolving the holography in the plane-wave limit of AdS/CFT correspondence"*, Nucl. Phys. B711, 3 (2005), hep-th/0406225. • A. Tsuji, *"Holography of Wilson loop correlator and spinning strings"*, Prog. Theor. Phys. 117, 557 (2007), hep-th/0606030.

[32] R. A. Janik, P. Surowka and A. Wereszczynski, *"On correlation functions of operators dual to classical spinning string states"*, JHEP 1005, 030 (2010), arxiv:1002.4613.

[33] E. I. Buchbinder and A. A. Tseytlin, *"On semiclassical approximation for correlators of closed string vertex operators in AdS/CFT"*, arxiv:1005.4516.

[34] K. Zarembo, *"Holographic three-point functions of semiclassical states"*, JHEP 1009, 030 (2010), arxiv:1008.1059. • M. S. Costa, R. Monteiro, J. E. Santos and D. Zoakos, *"On three-point correlation functions in the gauge/gravity duality"*, JHEP 1011, 141 (2010), arxiv:1008.1070.

[35] R. Roiban and A. Tseytlin, *"On semiclassical computation of 3-point functions of closed string vertex operators in AdS_5xS^5"*, Phys.Rev. D82, 106011 (2010), arxiv:1008.4921. • S. Ryang, *"Correlators of Vertex Operators for Circular Strings with Winding Numbers in $AdS5xS5$"*, JHEP 1101, 092 (2011), arxiv:1011.3573. • T. Klose and T. McLoughlin, *"A light-cone approach to three-point functions in AdS_5xS^5"*, arxiv:1106.0495. • S. Ryang, *"Extremal Correlator of Three Vertex Operators for Circular Winding Strings in AdS_5xS_5"*, JHEP 1111, 026 (2011), arxiv:1109.3242, * Temporary entry *.

[36] R. Hernandez, *"Three-point correlation functions from semiclassical circular strings"*, J. Phys. A44, 085403 (2011), arxiv:1011.0408. • J. G. Russo and A. A. Tseytlin, *"Large spin expansion of semiclassical 3-point correlators in AdS_5xS^5"*, JHEP 1102, 029 (2011), arxiv:1012.2760. • G. Georgiou, *"Two and three-point correlators of operators dual to folded string solutions at strong coupling"*, JHEP 1102, 046 (2011), arxiv:1011.5181. • C. Park and B.-H. Lee, *"Correlation functions of magnon and spike"*, arxiv:1012.3293. • D. Bak, B. Chen and J.-B. Wu, *"Holographic Correlation Functions for Open Strings and Branes"*, arxiv:1103.2024. • A. Bissi, C. Kristjansen, D. Young and K. Zoubos, *"Holographic three-point functions of giant gravitons"*, arxiv:1103.4079. • R. Hernandez, *"Three-point correlators for giant magnons"*, JHEP 1105, 123 (2011), arxiv:1104.1160. • C. Ahn and P. Bozhilov, *"Three-point Correlation functions of Giant magnons with finite size"*, arxiv:1105.3084. • D. Arnaudov, R. C. Rashkov and T. Vetsov, *"Three- and four-point correlators of operators dual to folded string solutions in AdS_5xS^5"*, arxiv:1103.6145. • C. Ahn and P. Bozhilov, *"Three-point Correlation Function of Giant Magnons in the Lunin-Maldacena background"*, arxiv:1106.5656.

[37] J. Escobedo, N. Gromov, A. Sever and P. Vieira, *"Tailoring Three-Point Functions and Integrability"*, JHEP 1109, 028 (2011), arxiv:1012.2475.

[38] J. Escobedo, N. Gromov, A. Sever and P. Vieira, *"Tailoring Three-Point Functions and Integrability II. Weak/strong coupling match"*, JHEP 1109, 029 (2011), arxiv:1104.5501.

[39] N. Gromov, A. Sever and P. Vieira, *"Tailoring Three-Point Functions and Integrability III. Classical Tunneling"*, arxiv:1111.2349, * Temporary entry *.

[40] A. Bissi, T. Harmark and M. Orselli, *"Holographic 3-point function at one loop"*, arxiv:1112.5075.

[41] G. Georgiou, *"$SL(2)$ sector: weak/strong coupling agreement of three-point correlators"*, JHEP 1109, 132 (2011), arxiv:1107.1850, * Temporary entry *.

[42] R. A. Janik and A. Wereszczynski, *"Correlation functions of three heavy operators: The AdS contribution"*, arxiv:1109.6262.

[43] E. Buchbinder and A. Tseytlin, *"Semiclassical correlators of three states with large S^5 charges in string theory in AdS_5xS^5"*, arxiv:1110.5621, * Temporary entry *.

[44] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *"A semi-classical limit of the gauge/string correspondence"*, Nucl. Phys. B636, 99 (2002), hep-th/0204051.

[45] Y. Kazama and S. Komatsu, *"On holographic three point functions for GKP strings from integrability"*, arxiv:1110.3949, * Temporary entry *.

[46] N. Drukker and J. Plefka, *"The structure of n-point functions of chiral primary operators in N=4 super Yang-Mills at one-loop"*, JHEP 0904, 001 (2009), arxiv:0812.3341.

[47] E. D'Hoker, D. Z. Freedman, S. D. Mathur, A. Matusis and L. Rastelli, *"Extremal correlators in the AdS/CFT correspondence"*, hep-th/9908160, in: *"The Many Faces of the Superworld"*, ed.: M. Shifman, World Scientific (1999), Singapore.

[48] G. Georgiou, V. Gili and J. Plefka, *"The two-loop dilatation operator of $N = 4$ super Yang-Mills theory in the $SO(6)$ sector"*, arxiv:1106.0724.

[49] A. Grossardt and J. Plefka, *"One-Loop Spectroscopy of Scalar Three-Point Functions in planar N=4 super Yang-Mills Theory"*, arxiv:1007.2356.

[50] G. Georgiou and G. Travaglini, *"Fermion BMN operators, the dilatation operator of N=4 SYM, and pp wave string interactions"*, JHEP 0404, 001 (2004), hep-th/0403188.

[51] A. Grossardt, *"Three-Point Functions in Superconformal Field Theories"*, Diploma thesis, Institute of Physics, Humboldt-University Berlin , (2010), http://qft.physik.hu-berlin.de.